Designing with Farey Fractions

In this day of hand-held calculators and computers, for most of us fractions are only dim, unpleasant memories of early rote schooling and seemingly pointless, tedious exercises.

Despite the fact that we can get along without all but the simplest fractions for most everyday business, fractions are important in mathematics, the physical sciences, and computer science.

Fractions may seem to be unlikely candidates for design inspiration, but patterns and beauty can be found in the most unexpected places in mathematics.

Farey Fractions

The Farey fractions, named after the British geologist John Farey (1766-1826), provide an example.

The Farey fraction sequence of order i, $\mathcal{F}(i)$, consists of all fractions with values between 0 and 1 whose denominators do not exceed i, expressed in lowest terms and arranged in order of increasing magnitude. For example, $\mathcal{F}(6)$ is

$$\frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{1}{1}$$

Farey observed that the fractions in such sequences are the *mediants* of their adjacent fractions. The mediant of n_1/d_1 and n_2/d_2 is

$$(n_1 + n_2)/(d_1 + d_2)$$

which looks like a naive attempt to add fractions.

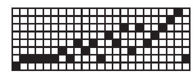
Farey sequences have a number of other interesting and useful properties [1, 2]. Our interest here, however, is with their use in weave design.

A sequence of fractions can be interpreted as integer sequences in a number of ways. Since the numerators and denominators show distinctive patterns, a natural method is to separate a sequence of fractions into two sequences, one of the numerators and one of the denominators as in:

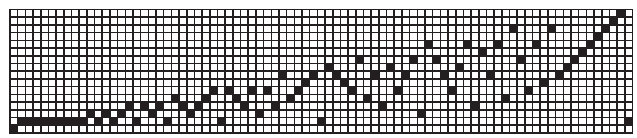
$$\mathcal{F}_n(6) = 0, 1, 1, 1, 1, 2, 1, 3, 2, 3, 4, 5, 1$$

$$\mathcal{F}_d(6) = 1, 6, 5, 4, 3, 5, 2, 5, 3, 4, 5, 6, 1$$

The patterns in Farey sequences can be seen in grid plots, as shown in Figures 1 and 2. The bottom line of a plot corresponds to the smallest value in the sequence.

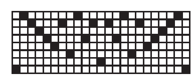


$\mathcal{F}_n$**(8)**

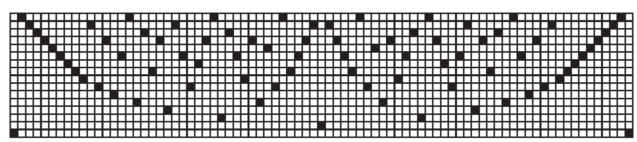


$\mathcal{F}_n$**(16)**

Figure 1. Farey Numerator Sequences



$\mathcal{F}_d$**(8)**



$\mathcal{F}_d$**(16)**

Figure 2. Farey Denominator Sequences

The patterns for other values of i are similar in structure. As i gets larger, the sequences are longer and the patterns more articulated.

The patterns in the numerator sequences are clear and interesting, although not particularly attractive. The patterns in the denominator sequences, however, are very attractive. Part of this is because these sequences are palindromic, adding the visual appeal of symmetry. A palindrome can be constructed from any sequence, but this one occurs naturally.

Properties of Farey Sequences

Farey sequences have several properties that relate to their appropriateness for weave design.

Both $\mathfrak{F}_n(i)$ and $\mathfrak{F}_d(i)$ contain i different values. The 0 in $\mathfrak{F}_n(i)$ can be handled in various ways. One way is to add 1 to all values in the sequence. Another way is to use modular shaft arithmetic [3] with modulus i, in which case the 0 is changed to i and all other values remain unchanged. In any event, all shafts and treadles in their ranges are utilized.

The distribution of values in the sequences is not balanced, however. The value 0 appears only once in $\mathfrak{F}_n(i)$ and the value 2 appears only once in $\mathfrak{F}_d(i)$ for $i > 1$ (for $i = 1$, 2 does not appear at all, but this is an uninteresting case for weave design). The distributions of other values follow interesting patterns, but that is a deeper topic that we won't consider here.

In $\mathfrak{F}_n(i)$ there is a string of 1s of length $\lfloor i/2 \rfloor + 1$ starting with the second value of the sequence, where $\lfloor x \rfloor$ is the integer part of x. No other successive values are the same. For $i > 1$, no successive values in $\mathfrak{F}_d(i)$ are the same.

The lengths of Farey sequences increase only modestly with i. There is no simple formula, but the length is about

$$3(i/\pi)^2 \approx 0.304 \times i^2$$

which gives increasingly better approximations as i gets larger [2].

Here are the lengths of $\mathfrak{F}(i)$ for $4 \le i \le 32$:

i	*length*	*i*	*length*
4	7	19	121
5	11	20	129
6	13	21	141
7	19	22	151
8	23	23	173
9	29	24	181
10	33	25	201
11	43	26	213
12	47	27	231
13	59	28	243
14	65	29	271
15	73	30	279
16	81	31	309
17	97	32	325
18	103		

Drafting with Farey Sequences

One way to use Farey sequences is directly as threading and treadling sequences. Figures 3 and 4 show drawdown patterns for $\mathfrak{F}(8)$ with 8 shafts and 8 treadles, treadled as drawn in. Direct tie-ups were used to make the patterns clear.

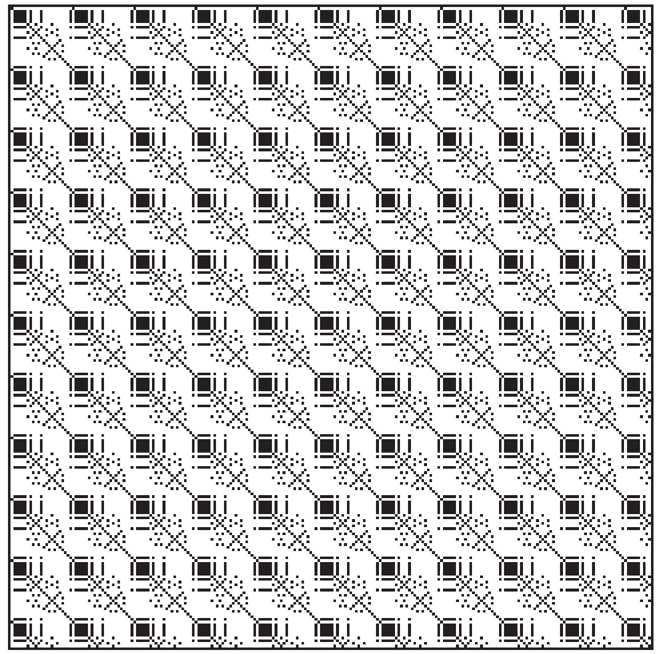


Figure 3. $\mathfrak{F}_n$**(8)**

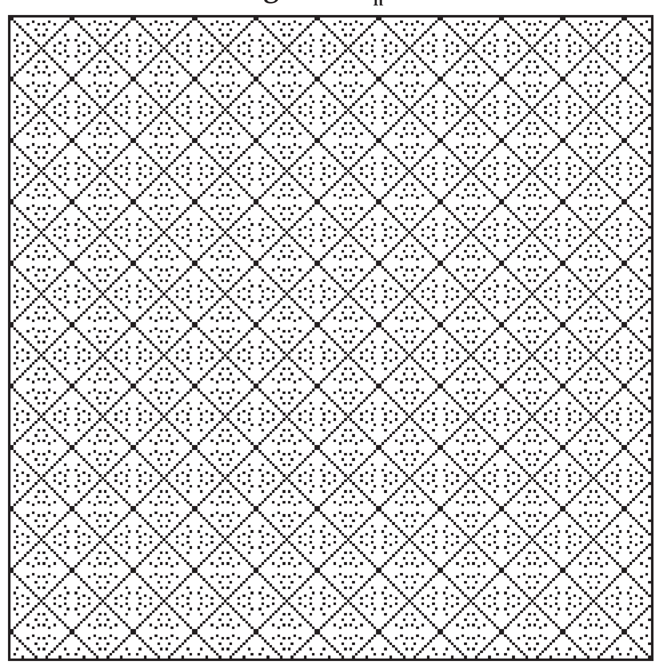


Figure 4. $\mathfrak{F}_d$**(8)**

Figure 5 shows the drawdown pattern for $\mathfrak{F}_n(8)$ threading and $\mathfrak{F}_d(8)$ treadling.

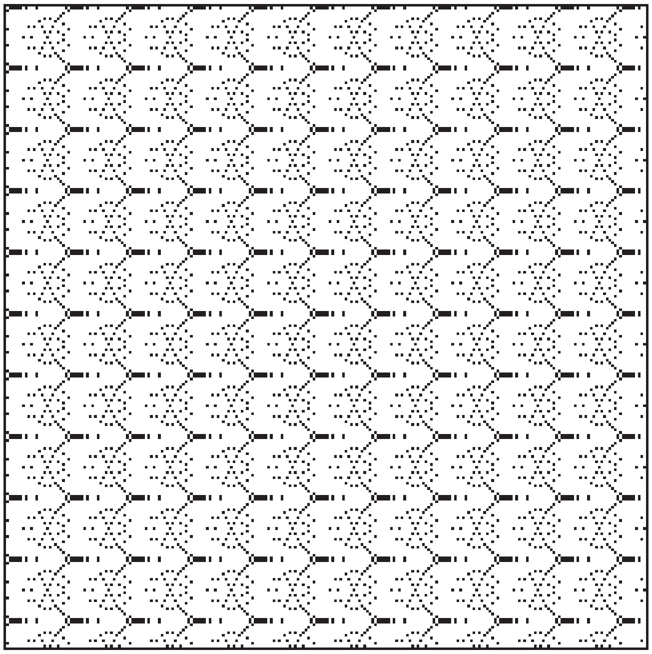


Figure 5. $\mathfrak{F}_n(8)$ versus $\mathfrak{F}_d(8)$

Direct tie-ups are not suitable for weaving with these sequences for structural reasons. Figures 6-8 show the corresponding drawdown patterns for $\frac{2}{\;\;2}$ twill tie-ups.

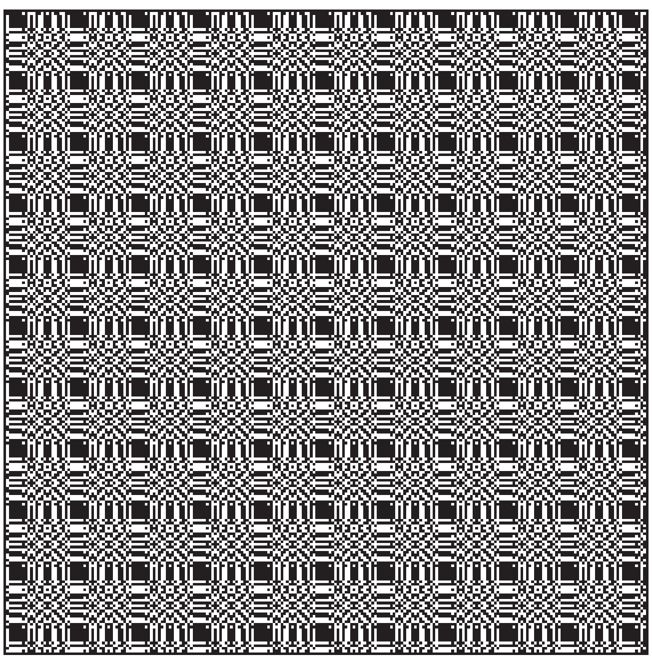


Figure 6. $\mathfrak{F}_n(8)$ Twill

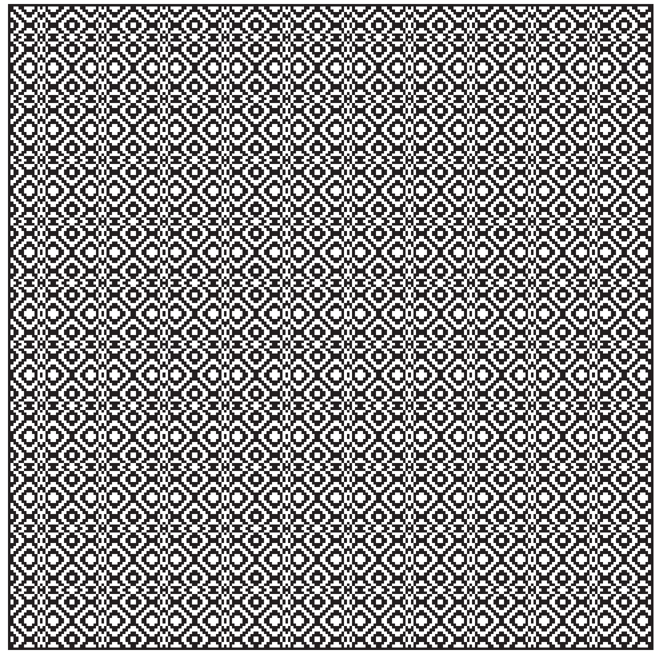


Figure 7. $\mathfrak{F}_d(8)$ Twill

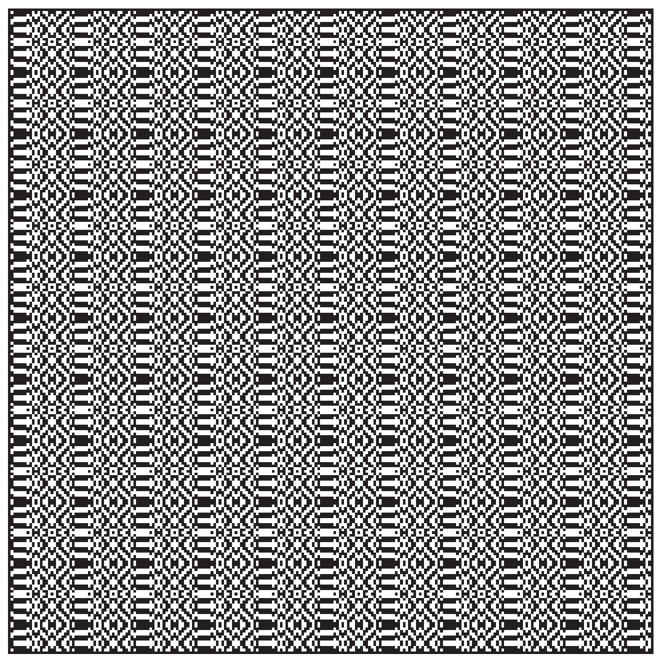


Figure 8. $\mathfrak{F}_n(8)$ versus $\mathfrak{F}_d(8)$ Twill

Adapting Farey Sequences for Thread-by-Thread Drafts

Modifications often are needed to make sequences from mathematical sources suitable for weaving or to improve the appearance of weaves derived from them.

This usually means doing some violence to the mathematical properties of the sequences, but weave design is, after all, an artistic enterprise — mathematics can only provide inspiration.

Numerator sequences are more troublesome than denominator sequences because numerator sequences have a string of 1s starting at term 2.

One thing to do is to simply remove successive duplicates. This is an easy method that can be applied to all sequences that have successive duplicate values.

Another method is to add incidentals between successive duplicates, analogous to the use of incidentals in name drafting to produce alternating odd/even values for overshot [4].

Yet another method is to change alternative values to break the sequence of duplicates. This has the virtue of maintaining the length of the sequence. For numerator sequences, an attractive method is to change every other 1 into a 0. See Figures 9 and 10.

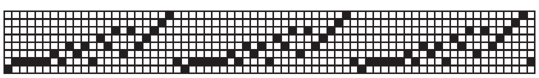


Figure 9. $\mathcal{F}_n(8)$ Repeated

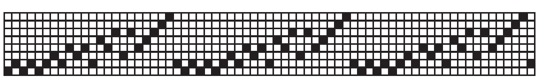


Figure 10. $\mathcal{F}_n(8)$ With Changes Repeated

The drawdown pattern for the changed sequence is shown in Figure 11. Compare this with Figure 3, which shows the pattern without changes.

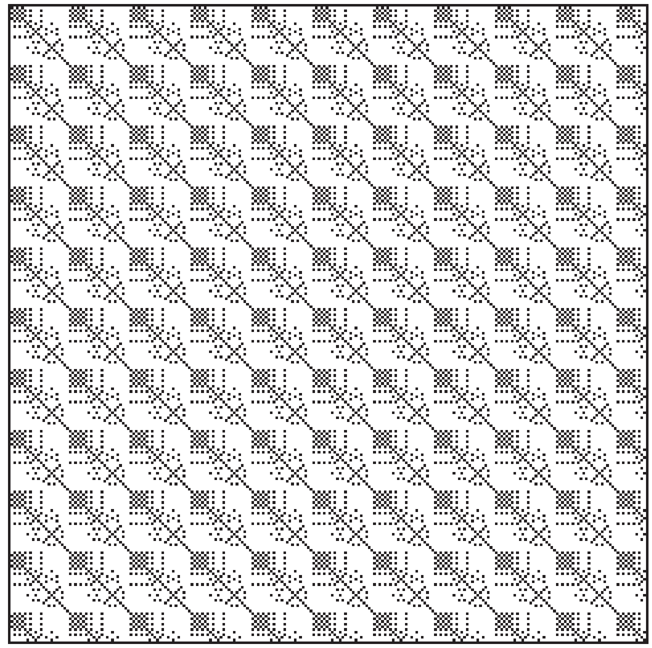


Figure 11. $\mathcal{F}_n(8)$ with Changes

When considering successive duplicates, it is important to look at the first and last values of a sequence, since these become adjacent when the sequence is repeated. Farey numerator sequences have first and last values of 0 and 1, respectively. Note that this meshes with the 0, 1 change method just discussed.

Farey denominator sequences are true palindromes with the same first and last value. For repeats, *pattern palindromes* obtained by removing the last value of a pure palindrome usually are used. Then, of course, a true palindrome for the entire pattern is obtained by appending the first value of the pattern palindrome to the end of the last repeat. The difference this makes in the drawdown pattern is minor. Compare Figure 12 with Figure 4.

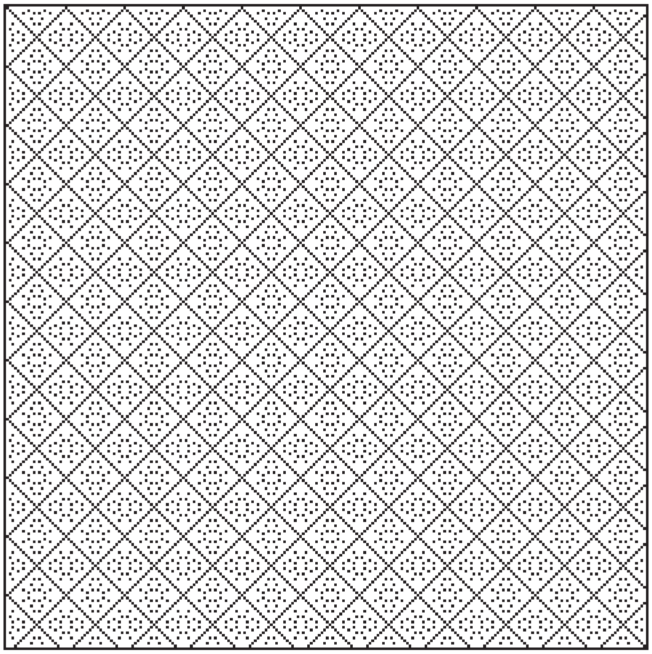


Figure 12. $\mathcal{F}_d(8)$ with Change

Drafting Variations

Parameters

Even with just two sequences used for thread-by-thread drafts, there are endless variations for drafting. The parameters are

- *i* Farey order for threading
- *j* Farey order for treadling
- *m* number of shafts, $\leq i$
- *n* number of treadles, $\leq j$
- *t* threading sequence type (numerator or denominator)
- *u* treadling sequence type (numerator or denominator)

One general question is what happens if $m < i$ and/or $n < j$, assuming modular arithmetic is used to reduce the sequences appropriately.

Figure 13 shows the pattern for $\mathcal{F}_d(8)$ threading and treadling with 4 shafts and 4 treadles.

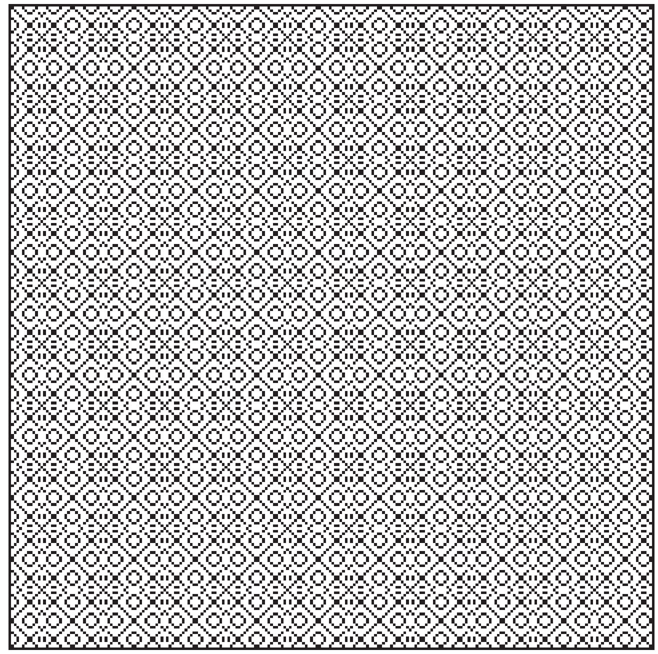


Figure 13. $\mathcal{F}_d(8)$ with 4 Shafts and 4 Treadles

Another question is what happens if *m* does not divide *i* evenly or if *n* does not divide *j* evenly. Endless questions and possibilities … .

Another possibility is to use a Farey sequence for threading and some unrelated sequence for treadling, or vice versa. Figure 14 shows the pattern for $\mathcal{F}_d(8)$ threading and an ascending straight draw for treadling.

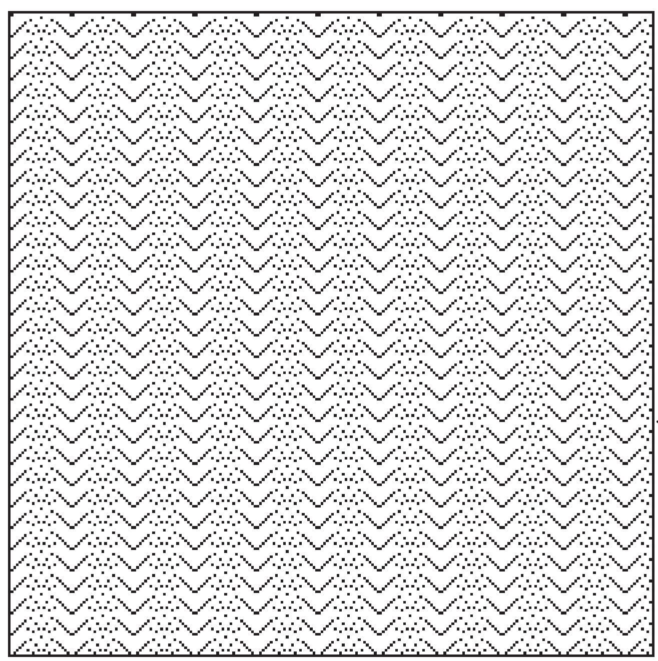


Figure 14. $\mathcal{F}_d(8)$ with Straight-Draw Treadling

And what about tie-ups? We've used direct tie-ups to show underlying interlacement patterns and a simple twill to show weavable interlacements. Are there other tie-ups that produce more attractive patterns?

Palindromes for Numerator Sequences

Farey denominator sequences are palindromic but Farey numerator sequences are not. Another design possibility is to form palindromes by reflecting numerator sequences and use them alone or in combination with denominator sequences.

Figure 15 shows the pattern for $\mathcal{F}_n(8)$, reflected and treadled as drawn in. Figure 16 shows the pattern for $\mathcal{F}_n(8)$, reflected for threading and $\mathcal{F}_d(8)$ for treadling.

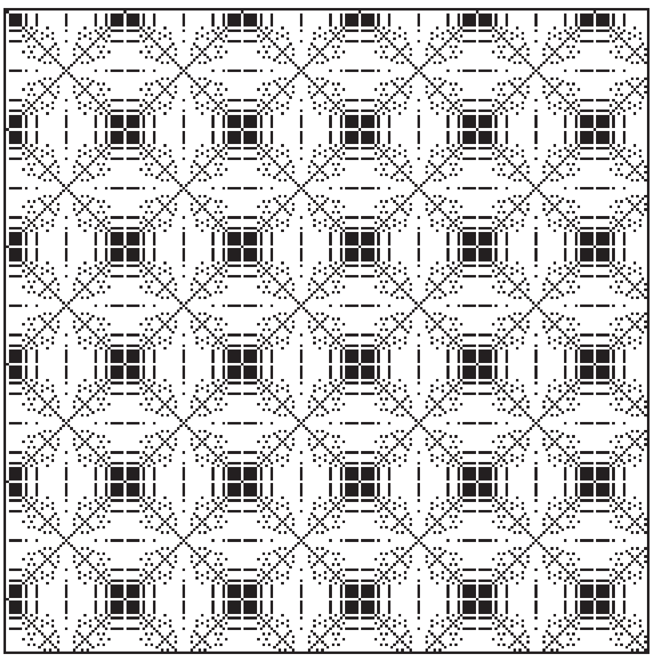


Figure 15. $\mathcal{F}_n(8)$ Reflected

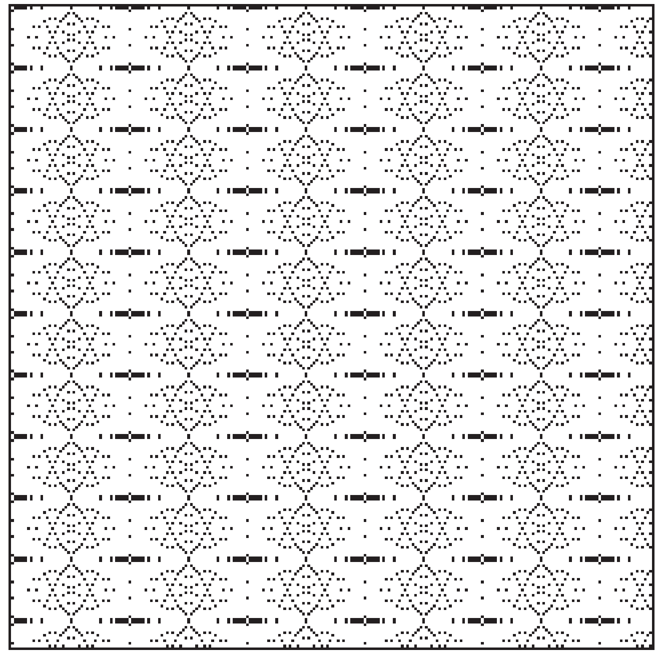


Figure 16. $\mathcal{F}_n(8)$ Reflected Threading Versus $\mathcal{F}_d(8)$

Interleaved Sequences

Another possibility is to interleave the numerator and denominator sequences. For $\mathcal{F}_n(6)$ the result is

0, 1, 1, 6, 1, 5, 1, 4, 1, 3, 2, 5, 1, 2, 3, 5, 2, 3, 3, 4, 4, 5, 5, 6, 1, 1

Figure 17 shows the pattern for $\mathcal{F}_n(8)$ and $\mathcal{F}_n(8)$ interleaved, treadled as drawn in with a direct tie-up. Figure 18 shows the pattern for the corresponding $\frac{2}{2}$ twill.

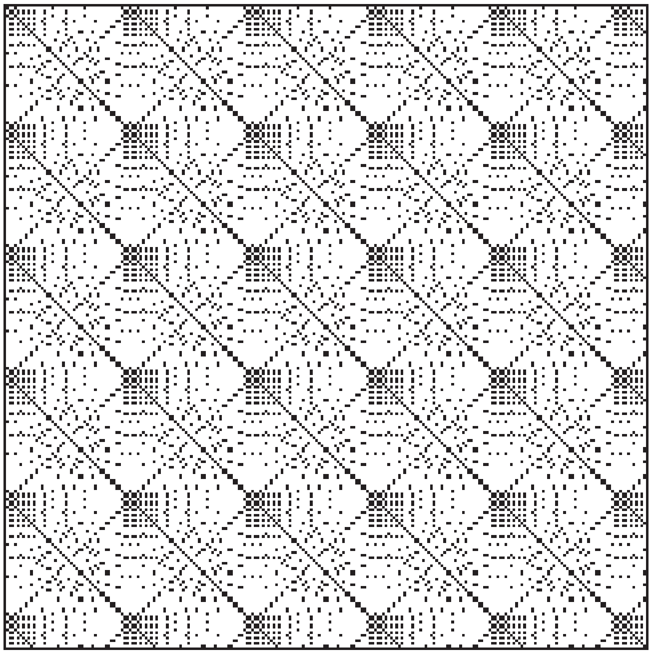


Figure 17. $\mathcal{F}_n(8)$ and $\mathcal{F}_d(8)$ Interleaved

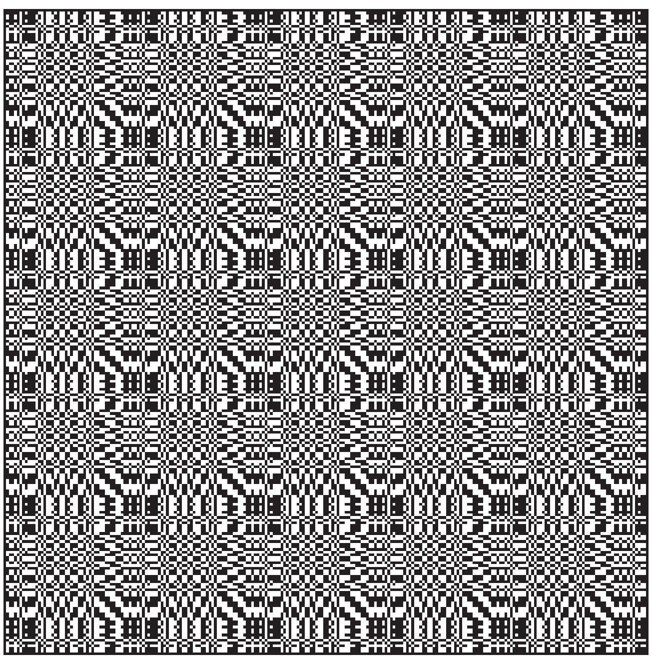


Figure 18. $\mathcal{F}_n(8)$ and $\mathcal{F}_d(8)$ Interleaved Twill

Combining Farey Sequences of Different Orders

So far we've only shown patterns based on Farey Sequences of the one order. Figure 19 shows the pattern that results of concatenating Farey denominator sequences of orders 1 through 8, treadled as drawn in with a direct tie-up.

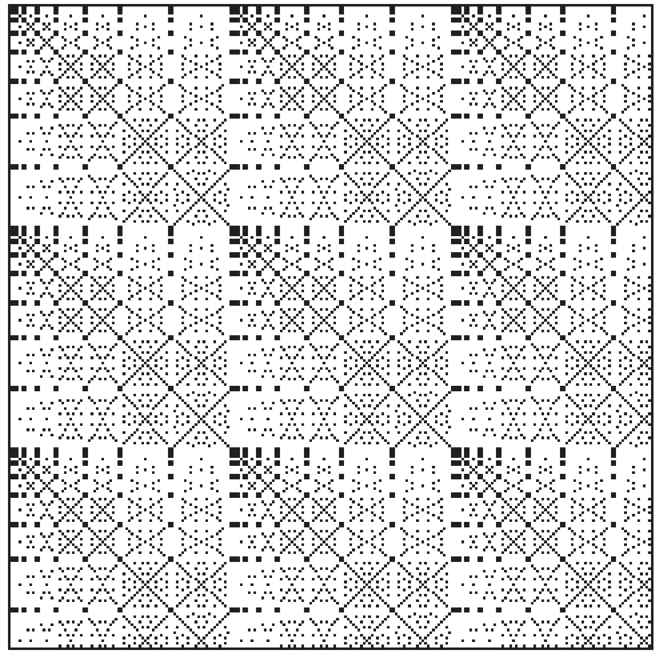


Figure 19. $\mathcal{F}_d(1)$, $\mathcal{F}_d(2)$, …, $\mathcal{F}_d(8)$

You can figure out the pattern if this sequence is reflected to form a pattern palindrome.

Point Twill

While the direct use of Farey sequences for threading and treadling produces interesting results, interpreting the values in the sequences as inflection points for point draws is more promising. In such an interpretation, only high and low values in runs are considered. For example,

1, 1, 2, 5, 3, 1

has the inflection points 1, 5, and 1.

Figure 20 shows grid plots for $\mathcal{F}_d(8)$ point draws.

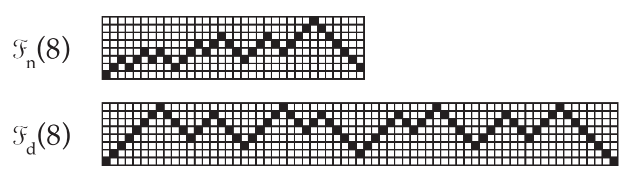


Figure 20. $\mathcal{F}(8)$ Point Draws

Figures 21 and 22 show the patterns for the corresponding $\frac{2}{\;2}$ point twills, treadled as drawn in.

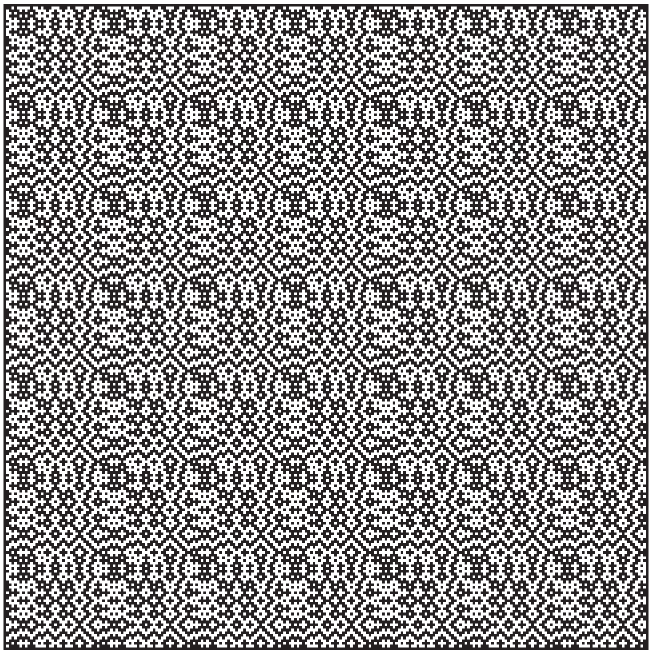


Figure 21. $\mathcal{F}_n(8)$ Point Twill

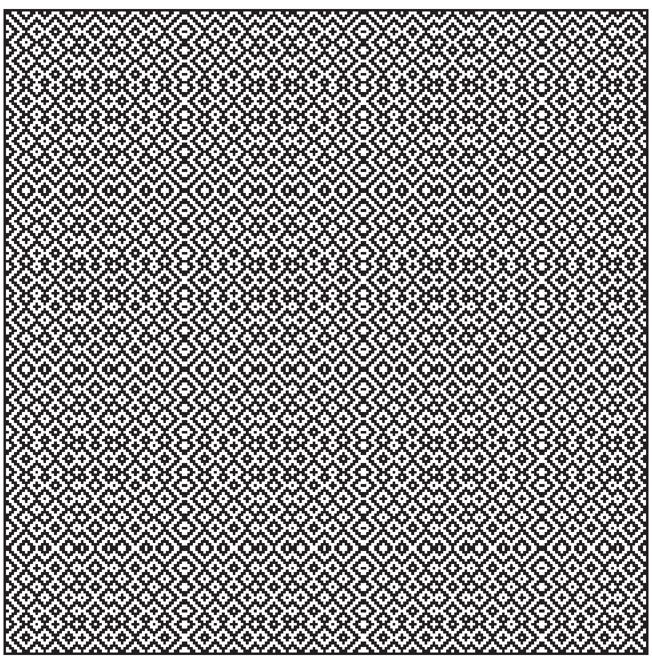


Figure 22. $\mathcal{F}_d(8)$ Point Twill

The lengths of point draws are, of course, considerably longer than the lengths of the sequences from which they are derived:

i	$\mathcal{F}_n(i)$ *point length*	$\mathcal{F}_d(i)$ *point length*
4	6	11
5	12	23
6	14	27
7	26	51
8	34	67
9	48	95
10	56	111
11	86	171
12	98	195
13	140	279
14	158	315
15	186	371
16	218	435
17	290	579
18	316	631
19	406	811
20	446	891
21	506	1011
22	556	1111
23	688	1375
24	736	1471
25	862	1723
26	934	1867
27	1056	2111
28	1140	2279
29	1350	2699
30	1414	2827
31	1654	3307
32	1782	3563

Other Possibilities

The wide range of possibilities touched on here does not begin to exhaust the potential of Farey fraction design.

Color specification, for example, always is a design possibility for sequences.

And, of course, Farey fraction design is not limited to weaving [5].

I'd be interested in your comments and suggestions. Send e-mail to ralph@cs.arizona.edu.

Resources

If you want to experiment with Farey sequences, you'll find sequence data at

http://www.cs.arizona.edu/patterns/weaving/sequences/fareynseq.html

and

http://www.cs.arizona.edu/patterns/weaving/sequences/fareydseq.html

References

1. *CRC Concise Encyclopedia of Mathematics*, Eric W. Weisstein, Chapman & Hall/CRC, 1999, pp. 674.

2. *The Book of Numbers*, John H. Conway and Richard K. Guy, Springer-Verlag, 1996, pp. 152-156.

3. *Residue Sequences in Weave Design*, Ralph E. Griswold, 2000: http://www.cs.arizona.edu/patterns/weaving/gre_res.pdf.

4. *A Handweaver's Notebook*, Heather G. Thorpe, Collier Books, 1956, pp. 153-156.

5. *The Farey Room*, Linas Vepstas: http://linas.org/art-gallery/farey/farey.html

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