The Morse-Thue Sequence

The Morse-Thue sequence is a binary fractal sequence with many interesting properties. It begins as

0, 1, 1, 0, 1, 0, 0, 1, …

This sequence was introduced in 1906 by the Norwegian mathematician Axel Thue (pronounced TOO) as an example of an aperiodic recursively computable string of symbols. Later Marvin Morse

> … proved that the trajectories of dynamic systems whose phase spaces have a negative curvature everywhere can be completely characterized by a *discrete* sequence of 0s and 1s — a stunning discovery [1].

(We quote because we don't understand it well enough to use our own words.)

Because of the importance of Morse's discovery, his name usually is listed first, although the sequence sometimes is called the Thue-Morse sequence.

Constructing the Morse-Thue Sequence

There are many ways of constructing this sequence. The one shown most often uses the *substitution map*

0 ➛ 01

1 ➛ 10

starting with the initial term 0. At each step every 0 and 1 is replaced by the specified pair, simultaneous (at least conceptually), Thus, the development proceeds like this:

0 ➛ 0, 1 ➛ 0, 1, 1, 0 ➛ 0, 1, 1, 0, 1, 0, 0, 1 ➛ …

Another way to produce the Morse-Thue sequence is to start with 0 and iterate the following process: Take the present sequence and append its complement to it. (By complementing, we mean replacing 0 by 1 and 1 by 0.)

It goes like this:

0

0, 1

0, 1, 1, 0

0, 1, 1, 0, 1, 0, 0, 1

0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0

…

The advantage of this method is that it is simple and the number of terms produced increases rapidly.

A third method for producing the Morse-Thue sequence is to write the nonnegative integers in binary form:

0, 1, 10, 11, 100, 101, 110, 111, …

Then replace every value by its digit reduction mod 2. Digit reduction sums the digits of a number and repeats the process if necessary until only one digit remains. Thus the digit reduction of 111 is 3, whose residue mod 2 is 1.

Properties of the Morse-Thue Sequence

The Morse-Thue sequence is self similar, as can be seen by striking out every even-numbered value, which produces the original sequence:

0, ~~1~~, 1, ~~0~~, 1, ~~0~~, 0, ~~1~~, 1, ~~0~~, 0, ~~1~~, 0, ~~1~~, 1, ~~0~~, …

Among the fascinating properties of the Morse-Thue sequence is that it is *cube-free*. This means that it does not contain the subsequences 0, 0, 0 or 1, 1, 1. But cube-free is a more general concept. In the jargon of combinatorics on words [2], a word is any sequence of characters from the alphabet being used (here, 0 and 1). Cube-free applies to all words. For example, if

W = 1, 0, 1, 1, 0

(which is a word in the Morse-Thue sequence), then W, W, W, or

1, 0, 1, 1, 0, 1, 0, 1, 1, 0, 1, 0, 1, 1, 0

does not occur in the Morse-Thue sequence.

Generalizing the Morse-Thue Sequence

The Morse-Thue sequence generalizes to bases other than 2. For example, the base-5 generalization of the Morse-Thue sequence is

0, 1, 2, 3, 4, 1, 2, 3, 4, 0, 2, 3, 4, 0, 1, 3, 4, 0,
1, 2, 4, 0, 1, 2, 3, 1, 2, 3, 4, 0, …

All three methods used for computing the regular, base-2 Morse-Thue sequence generalize for larger bases.

Geometrical Interpretations of the Morse-Thue Sequence

If we visualize 0 as a black square and 1 as a white square, the Morse-Thue sequence appears graphically as shown in Figure 1:

Figure 1. The Morse-Thue Sequence

The steps for the append-complement method of construction are shown in Figure 2.

Figure 2. Morse-Thue Sequence Construction

This can be extended to two dimensions by at each step appending the complement both horizontally and vertically [3]. Figure 3 shows the first four iterations:

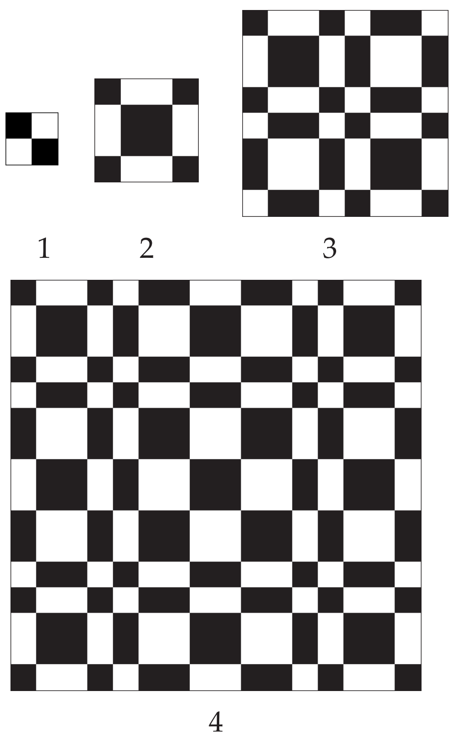

Figure 3. Constructing The Morse-Thue Plane

Like the Morse-Thue sequence itself, the Morse-Thue plane is fractal. And, despite the appearance of symmetry and regularity, there are no repetitions. That is, no finite portion of the plane can be tiled regularly to produce the whole plane.

The geometric interpretation of the Morse-Thue sequence extends to higher dimensions. Figure 4 shows a Morse-Thue cube.

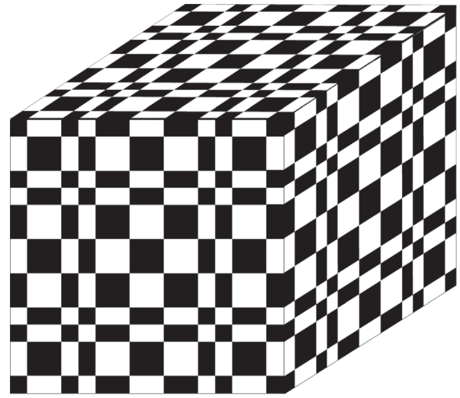

Figure 4. A Morse-Thue Cube

Try visualizing a Morse-Thue tesseract and higher-dimensional Morse-Thue cubes.

Applications of the Morse-Thue Sequence

The Morse-Thue sequence has applications in many areas. In addition to the one mentioned at the beginning of this article, the Morse-Thue sequence has been used in graphic design and in music composition [4-6].

It should not be surprising to discover that the Morse-Thue sequence can be used as the basis for a variety of interesting weaves. Figure 5 shows a weaving draft that was "drawn up" from the sixth iteration of the plane-construction process shown in Figure 3. Notice that the Morse-Thue sequence appears in the threading and treadling and that it takes only two shafts and two treadles to produce this weave.

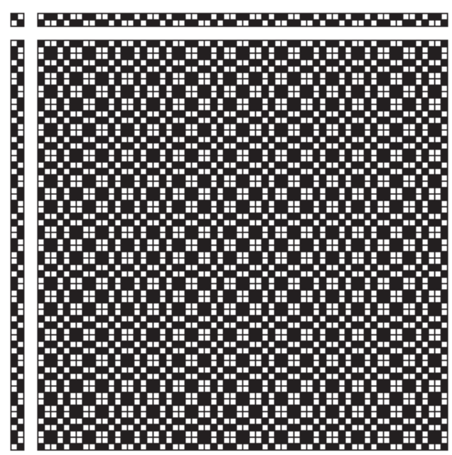

Figure 5. A Morse-Thue Weave

There are other ways the Morse-Thue sequence can be used in weave design. We'll explore some of these in subsequent articles.

Closing Comment

The Fibonacci sequence is the only sequence that has more interesting, almost magical properties than the Morse-Thue sequence. Perhaps we should bring the two together.

References

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