Design Inspirations from Multivariate Polynomials, Part 2

Review

In the first of this series of articles [1], we described Ada Dietz's method of weave design using powers of multivariate polynomials with special properties: unit coefficients and all variables to the first power:

$$(a_1 + a_2 + \ldots + a_n)^k$$

In this article, we'll consider other kinds of polynomials and various operations on polynomials.

Other Kinds of Multivariate Polynomials

Design sequences produced from Dietz polynomials have a very regular, rigid form. Every variable (representing, say, a block) appears in the sequence in the same way.

Other kinds of polynomials give more varied results. For example,

$$(a + 2b^2 + 3c^3)^2$$

produces the design sequence

aaabbabbabbabbacccacccacccacccacccacccbbbbbbbbb
bbbbbbbcccbbcccbbcccbbcccbbcccbbcccbbcccbbcccbb
cccbbcccbbcccbbccccccccccccccccccccccccccccccccc
cccccccccccccccccccc

Assigning *a*, *b*, and *c* to shafts 1, 2, and 3, respectively, and with a direct tie-up, treadled as drawn in, the drawdown pattern shown in Figure 1 results.

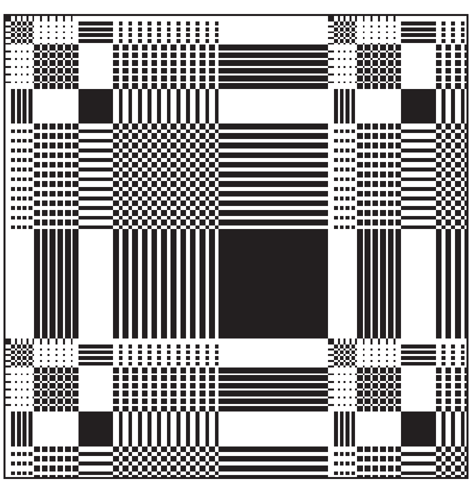

Figure 1. $(a + 2b^2 + 3c^3)^2$

Raising a polynomial to a power is what gives the design sequence richness and variety. Raising to a power is, of course, multiplying a polynomial by itself. Another possibility is to multiply different polynomials together, as in

$$(a + 2b^2 + 3c^3) \times (3a^2 + b^3 + 2c)$$

which produces the design sequence

aaaaaaaaaaabbaabbaabbaabbaabbaabbaacccaacccaaccc
aacccaacccaacccaacccaacccaacccabbbabbbabbbacacaca
cacacbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbcccbbbccc
bbbcccbbbcccbbbcccbbbcccbbbcccbbbcccbbbcccbbcbbc
bbcbbcbbcbbcbbcbbcbbcbbcbbcbbccccccccccccccccccc
cccccccccccccccccccccccccccccccccccccccccccccccc
cc

and the drawdown pattern shown in Figure 2.

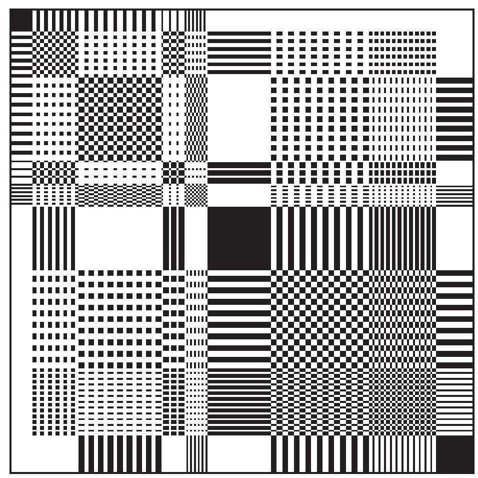

Figure 2. $(a + 2b^2 + 3c^3) \times (3a^2 + b^3 + 2c)$

And, of course, polynomials can be added, as in

$$(a + 3b^2 + 2c^2)^2 + (4a^2 + b^3 + c^3)$$

which produces the design sequence

aaaaaaaaaaabbabbabbabbabbabbaccaccaccaccbbbbbbb
bbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbcc
bbccbbccbbccbbccbbccbbccbbccbbccbbccbbccbbcccccc
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and the drawdown pattern shown in Figure 3.

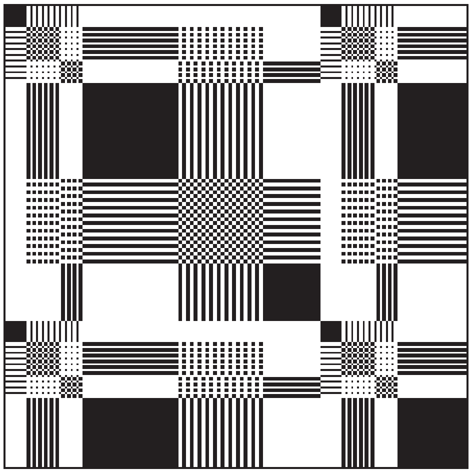

Figure 3. $(a + 3b^2 + 2c^2)^2 + (4a^2 + b^3 + c^3)$

We also can subtract polynomials, as well as allow negative coefficients. But this leads us into new territory.

The Domain of Coefficients

So far, all the coefficients have been positive integers. But negative coefficients, as in

$$(a - b + c)^2 \times (a - c + 2d)^3$$

are natural, as is the subtraction of polynomials, as in

$$(a + 3b^2 + 2c^2)^2 - (4a^2 + b^3 + c^3)$$

If there are negative terms, two new things may happen when the polynomial is multiplied out: (1) terms may be cancelled out by subtraction, and (2) there may be negative coefficients.

The cancellation of terms may be used to shorten strings produced by repetition. For example,

$$(a + b)^2 - ab = a^2 + ab + b^2 \rightarrow aaabbb$$

If there are negative coefficients, the question is how to interpret them in the concatenation step. There is no inverse to concatenation like subtraction is to addition. One option is to discard terms with negative coefficients. Another is to ignore the signs in the concatenation step, which is equivalent to using the absolute values of coefficients.

Neither of these alternatives makes any sense mathematically, but more important, they do not add anything to design possibilities. Instead, there is an opportunity here to add an additional degree of control in the construction of design sequences — if a coefficient is negative, reverse the subsequent sequence of variables before repeating it; the sign of a term determines the direction. For example,

$$a^2 - 3ab + b^2 \rightarrow aabababab b$$

Of course, reversal makes no sense mathematically either, since

$$3ab - 3ab = 0 \rightarrow \text{[nothing]}$$

while

$$3ab - 3ab \rightarrow abababbababa$$

But we are not "doing" mathematics. We're using mathematics as a basis for artistic design. Why not have a little nonsense?

Why restrict coefficients to integers? Why not fractions? But in producing design sequences, what are we to make of fractional coefficients in the concatenation step? One way would be to take their integer part, so that

$$\frac{1}{2} \rightarrow 0$$

and

$$\frac{3}{2} \rightarrow 1$$

An alternative that enriches design potential is to consider a fractional repetition as producing a corresponding fraction of the string of variables being repeated. Examples are

$$\frac{1}{2}ab \rightarrow a$$

$$-\frac{1}{2}ab \rightarrow b$$

$$\frac{3}{2}ab \rightarrow aba$$

If a fraction of a string of variables doesn't come out even, we can take just as many as it encompasses, as in

$$\frac{1}{2}aaabb \rightarrow aa$$

And why not admit all real numbers, so that

$$\sqrt{2}\,abcde \approx 1.414abcde \rightarrow abcdeab$$

and

$$\pi abcde \approx 3.1416 abcdefgh \rightarrow abcdefghabcdefghabcdefgha$$

Operations on Polynomials

So far we've only considered the addition, subtraction, and multiplication of polynomials. There are two other "elementary" operations on polynomials: Division and modular reduction.

Division

The division of two multivariate polynomials is carried out with respect to one variable, the other variables being treats as constants. If $n(a)$ and $d(a)$ are two polynomials in a and $d(a) \neq 0$, then there is a unique representation

$$\frac{n(a)}{d(a)} = q(a) + \frac{r(a)}{d(a)}$$

where $q(a)$ is the quotient polynomial and $r(a)$ is the remainder polynomial, and the degree of $r(a)$ in a is less than the degree of a in $n(a)$.

Polynomial division is not difficult and is much like long division. Consider this simple example:

$$\begin{array}{r l}
 & a^2 - ab + 3b^2 \\
a + b \,\big) & a^3 + 2ab^2 + bc^3 \\
 & \underline{a^3 + a^2b} \\
 & \quad -a^2b + 2ab^2 + bc^3 \\
 & \quad \underline{-a^2b - ab^2} \\
 & \qquad\quad 3ab^2 + bc^3 \\
 & \qquad\quad \underline{3ab^2 + 3b^3} \\
 & \qquad\qquad\quad bc^3 - 3b^3
\end{array}$$

Therefore, putting terms in order, $q(a) = a^2 - ab + 3b^2$ and $r(a) = -3b^3 + bc^3$.

To see what may happen in design, consider the Dietz polynomial $(a + b + c)^3$ divided by $a + 2b$. In this case the quotient and remainder polynomials are

$$q(a) = a^2 + 2ab + 3ac + b^2 + 3c^2$$

$$r(a) = -b^3 + 3b^2c - 3bc^2 + c^3$$

The drawdown patterns for this Dietz polynomial and the quotient and remainder are shown in Figures 4-6.

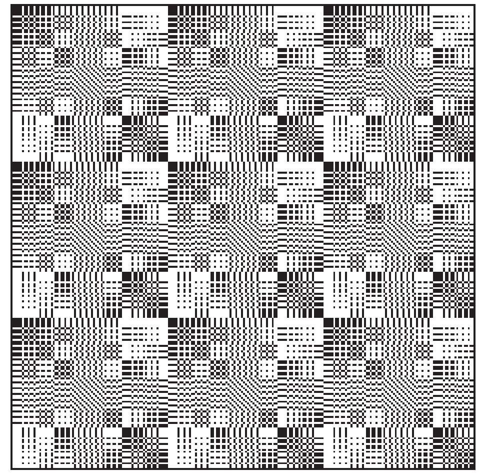

Figure 4. $(a + b + c)^3$

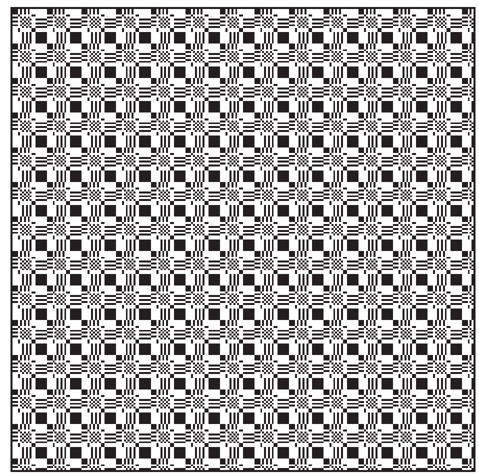

Figure 5. Quotient of $(a + b + c)^3 / (a + 2b)$

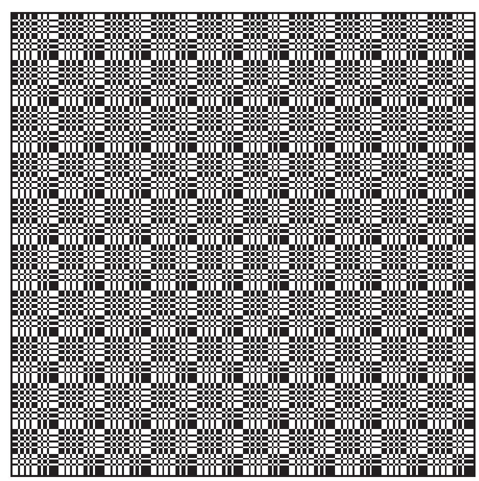

Figure 6. Remainder of $(a + b + c)^3 / (a + 2b)$

Modular Reduction

Modular reduction of a polynomial by a constant is simple: The coefficients of individual terms are take modulo the constant. Consider, for example, the example of adding two polynomials given earlier

$$(a + 3b^2 + 2c^2)^2 + (4a^2 + b^3 + c^3)$$

When multiplied out, the result is

$$5a^2 + 6ab^2 + 4ac^2 + 4b^3 + 9b^4 + 12b^2c^2 + 4c^3 + 4c^4$$

and the design polynomial is made up from the terms as follows

$5a^2$	*aaaaaaaaaa*
$6ab^2$	*abbabbabbabbabbabb*
$4ac^2$	*accaccaccacc*
$4b^3$	*bbbbbbbbbbbb*
$9b^4$	*bbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbb*
$12b^2c^2$	*bbccbbccbbccbbccbbccbbccbbccbbccbbccbbccbbccbbcc*
$4c^3$	*cccccccccccc*
$4c^4$	*cccccccccccccccc*

Concatenation gives

aaaaaaaaaaabbabbabbabbabbabbaccaccaccaccbbbbbbb
bbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbbccb
bccbbccbbccbbccbbccbbccbbccbbccbbccbbccbbcccccccc
cccccccccccccccccccc

Now consider this polynomial modulo 8:

$$5a^2 + 6ab^2 + 4ac^2 + 4b^3 + b^4 + 4b^2c^2 + 4c^3 + 4c^4$$

Only the coefficients 9 and 12 are affected, the other being less than 8. The terms for the design polynomial now are

$5a^2$	*aaaaaaaaaa*
$6ab^2$	*abbabbabbabbabbabb*
$4ac^2$	*accaccaccacc*
$4b^3$	*bbbbbbbbbbbb*
b^4	*bbbb*
$4b^2c^2$	*bbccbbccbbccbbcc*
$4c^3$	*cccccccccccc*
$4c^4$	*cccccccccccccccc*

and the design polynomial is:

aaaaaaaaaaabbabbabbabbabbabbaccaccaccaccbbbbbbb
bbbbbbbbbbccbbccbbccbbcccccccccccccccccccccccccc
cccc

The resulting drawdown pattern is shown in Figure 7.

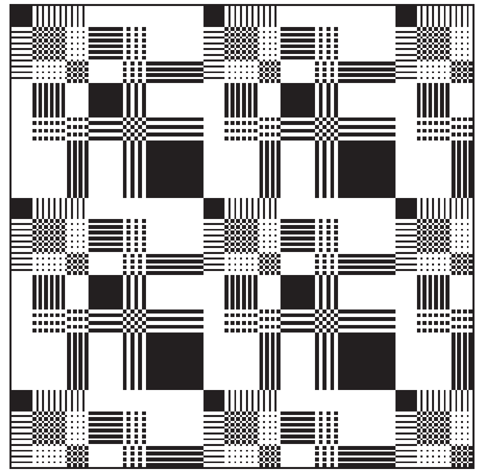

Figure 7. $((a + 3b^2 + 2c^2)^2 + (4a^2 + b^3 + c^3)) \equiv 8$

Modular reduction is a way of shortening otherwise overly long design sequences and thus provides interesting design possibilities.

When performing modular reduction on a polynomial, there is no affect unless the modulus is less than one of the coefficients. For example, moduli 12 and greater have no affect on the polynomial in the example above. When the modulus is the same as a coefficient, the term vanishes and makes no contribution to the design sequence.

As the modulus decreases, the maximum possible polynomial coefficient value decreases, shortening the design sequence. Little is left for modulus 2. See Appendix A.

A variety of other operations can be performed on multivariate polynomials to produce new polynomials. Differentiation and integration – derivatives and integrals — are the most obvious possibilities.

Differentiation

Differentiation is performed with respect to a specified variable and reduces the power of the polynomial in that variable.

The partial derivative with respect to one variable treats other variables as constants. The partial derivative of a polynomial p with respect to the variable a, which is denoted by $\partial(p)/\partial a$, requires application of only a few simple rules:

$$\partial a^n / \partial a = na^{n-1}$$
$$\partial(p + q) / \partial a = \partial p / \partial a + \partial q / \partial a$$
$$\partial \mathrm{k} / \partial a = 0$$

where k is a constant (indicated by roman type style rather than italic style); that is, a term not containing *a*. This includes *bc*, for example.

For example, if

$$p = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4$$

then

$$\partial p/\partial a = 5a^4 + 20a^3b + 30a^2b^2 + 20ab^3 + 5b^4$$

and

$$\partial p/\partial b = 5a^4 + 20a^3b + 30a^2b^2 + 20ab^3$$

The derivative of a polynomial can, of course, be differentiated. The second derivative of *p* with respect to *a*

$$\partial(\partial p/ \partial a)/\partial a$$

is abbreviated as

$$\partial^2 p / \partial a^2$$

and so on. For example,

$$\partial p^2 / \partial a^2 = 20a^3 + 60a^2b + 60ab^2 + 20b^3$$

and

$$\partial^3 p / \partial a^3 = 60a^{2+} 120ab + 60b^2$$

If differentiation continues with the same variable, the powers in the variable decrease, terms without an instance of the variable of differentiation vanish, and the result eventually becomes a constant, which does not yield a design sequence. See Appendix A.

Integration

Indefinite integration of an expression *p* with respect to a variable *a*, denoted by $\int p dx$, is the (approximate) inverse of differentiation. It is approximate because

$$\partial \mathrm{k} / \partial a = 0$$

regardless of the value of k. For example,

$$\partial(a^2 + 5) / \partial a = 2a$$

and

$$\partial(a^2 + 101) / \partial a = 2a$$

Therefore

$$\int 2a da = a^2 + \mathrm{k}$$

but k can have any value. This indeterminate k is called the constant of integration.

The constant of integration makes a difference in subsequent integrations, as in

$$\int\int 2a(da)^2 = \int(a^2 + \mathrm{k})\, da = \frac{1}{3}a^3 + \mathrm{k}a + \mathrm{j}$$

and then

$$\int\int\int 2a(da)^3 = \int\int(a^2 + \mathrm{k})\ (da)^2 = \int(\frac{1}{3}a^3 + \mathrm{k}a + \mathrm{j})da$$
$$= \frac{1}{12}a^4 + \frac{1}{2}\mathrm{k}a^2 + \mathrm{j}a + \mathrm{i}$$

and so on.

In constructing design polynomials using integration, this problem could be handled in several ways. One is to take all constants of integration to be 0, in which case we have

$$\int a da = \frac{1}{2}a^2$$

$$\int\int a(da)^2 = \frac{1}{6}a^3$$

$$\int\int\int a(da)^3 = \frac{1}{24}a^4$$

Another way is to take constants of integration to be parameters of the result, producing different polynomials depending on the values assigned to these constants. This opens the door to many possibilities but also increases the complexity of the problem. We'll take the constants of integration to be zero for the examples here.

If the constants of integration are taken to be zero and integration is repeatedly applied with the same variable of integration, the powers of the variable increase correspondingly but the coefficients become smaller and smaller until eventually the design sequence becomes empty. This happens very quickly for Dietz polynomials, which have unit coefficients. See Appendix C.

Conclusions

We have just touched on the possibilities for design based on multivariate polynomials and operations on them.

Any one of the subjects mentioned here could be extended into a substantial studt on its own.

One question, of course, is why? The now trite answer is because they are there. A better question, for which there is no present answer, is how polynomial structure is related to the visual interest and attractiveness of weaves based on their design sequences. A bit of mysticism may help here — numerology has its uses.

Reference

1. *Design Inspirations from Multivariate Polynomials*, Ralph E. Griswold, 2001:

http://www.cs.arizona.edu/patterns/weaving/webdocs/gre_pol1.pdf.

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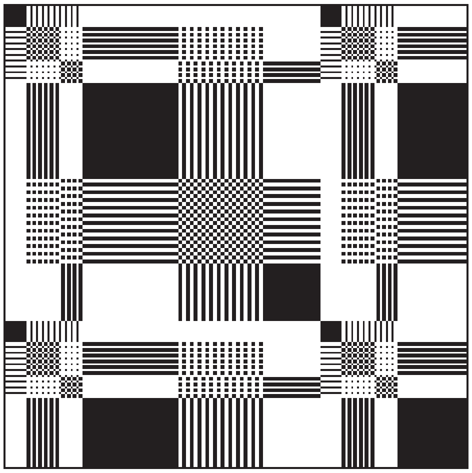
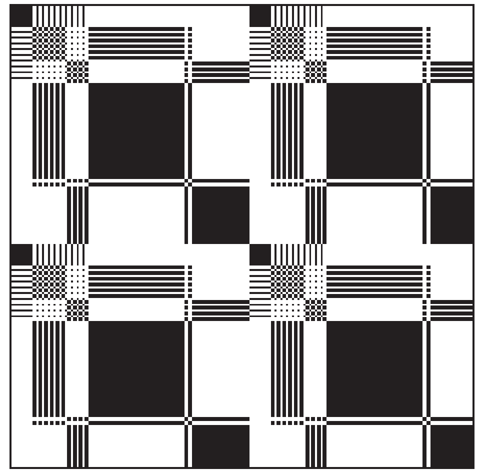
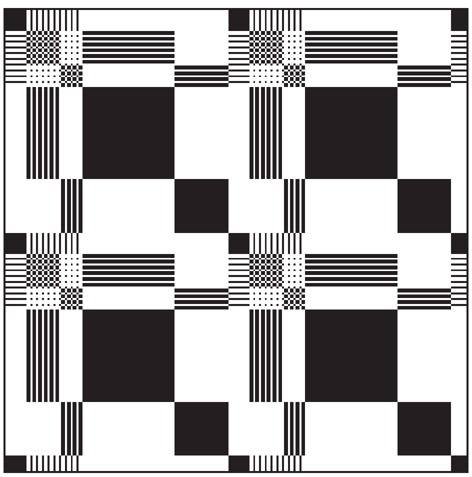
Appendix A — Modular Reduction



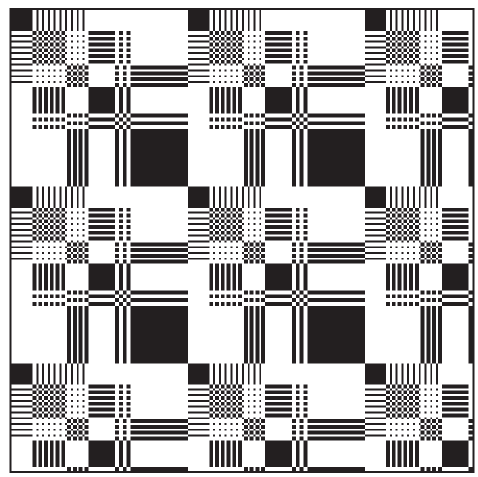
$((a + 3b^2 + 2c^2)^2 + (4a^2 + b^3 + c^3))$



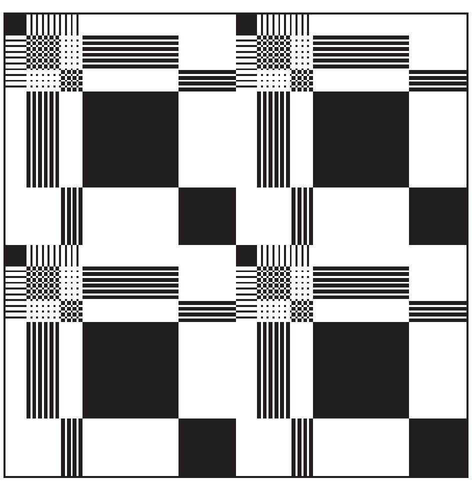
mod 10



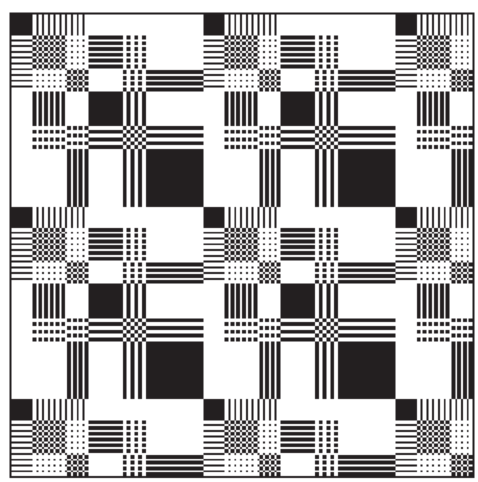
mod 12



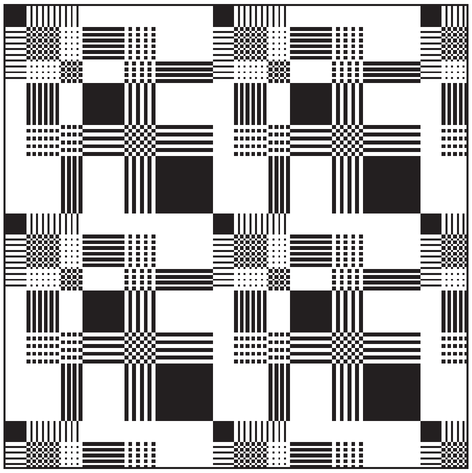
mod 9



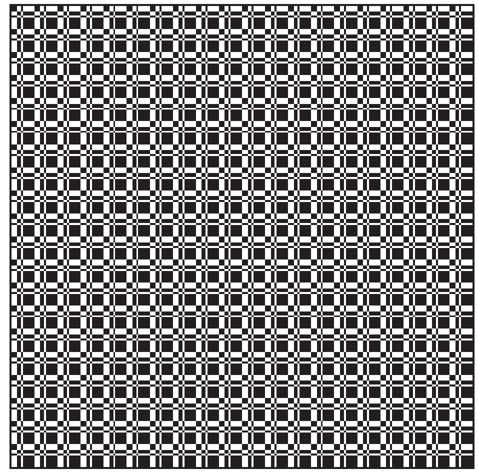
mod 11



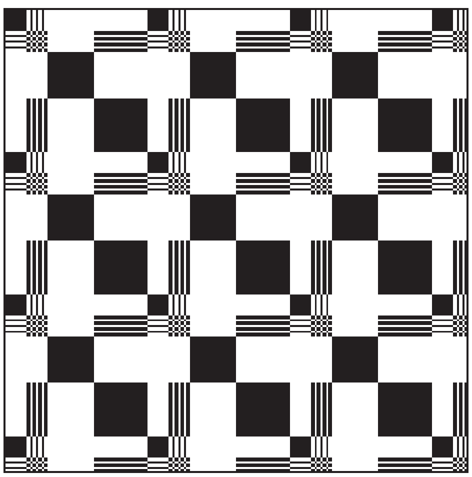
mod 8



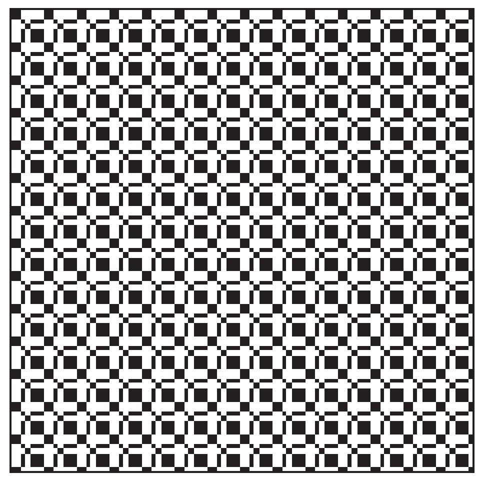
mod 7



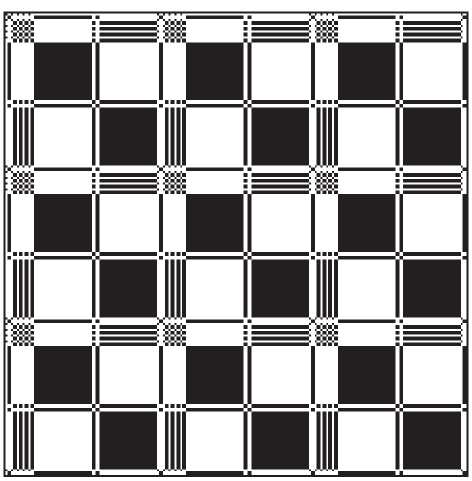
mod 4



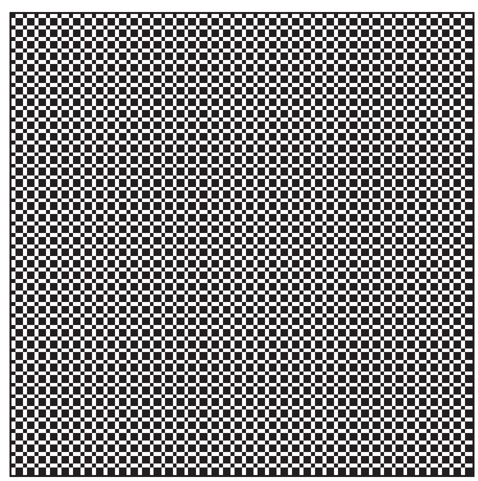
mod 6



mod 3



mod 5



mod 2

Appendix B — Differentiation

$(a + b + c)^4 = a^4 + 4a^3b + 4ac^3 + 6a^2b^2 + 12a^2bc + 6a^2c^2 + 4ab^3 + 12ab^2c + 12abc^2 + 4bc^3 + b^4 + 4b^3c + 6b^2c^2 + c^4$

aaaaaaabaaabaaabaaabaaacaaacaaacaaacaabbaabbaabbaabbaabbaabbaabcaabcaabcaabcaabcaabcaabcaabcaabcaabcaabcaabcaa
ccaaccaaccaaccaaccaaccabbbabbbabbbabbbabbcabbcabbcabbcabbcabbcabbcabbcabbcabbcabbcabbcabccabccabccabccabccabcca
bccabccabccabccabccabccacccacccacccacccbbbbbbbcbbbcbbbcbbbcbbccbbccbbccbbccbbccbbccbcccbcccbcccbccccccc

$\partial((a + b + c)^4)/\partial a = 4a^3 + 12a^2b + 12a^2c + 12ab^2 + 24abc + 12ac^2 + 4b^3 + 12b^2c + 12bc^2 + 4c^3$

aaaaaaaaaaaaaabaabaabaabaabaabaabaabaabaabaabaabaacaacaacaacaacaacaacaacaacaacaacaacabbabbabbabbabbabbabbabbabbab
babbabbabcabcabcabcabcabcabcabcabcabcabcabcabcabcabcabcabcabcabcabcabcabcabcabcabcaccaccaccaccaccaccaccaccaccacc
accaccbbbbbbbbbbbbbbcbbcbbcbbcbbcbbcbbcbbcbbcbbcbbcbbcbccbccbccbccbccbccbccbccbccbccbccbcccccccccccccc

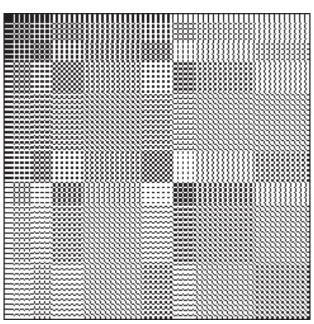
$\partial^2((a + b + c)^4)/(\partial a)^2 = 12a^2 + 24ab + 24ac + 12b^2 + 24bc + 12c^2$

aaaaaaaaaaaaaaaaaaaaaaaaabababababababababababababababababababababababacacacacacacacacacacacacacacacacacacacac
acacacacacbbbbbbbbbbbbbbbbbbbbbbbbbcbcbcbcbcbcbcbcbcbcbcbcbcbcbcbcbcbcbcbcbcbcbcbcbccccccccccccccccccccccc

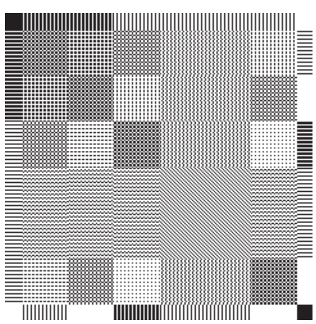
$\partial^3((a + b + c)^4)/(\partial a)^3 = 24a + 24b + 24c$

aaaaaaaaaaaaaaaaaaaaaaaabbbbbbbbbbbbbbbbbbbbbbbbcccccccccccccccccccccccc

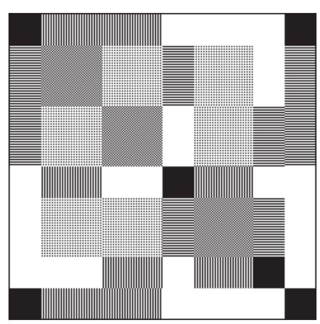
$\partial^4((a + b + c)^4)/(\partial a)^4 = 24$



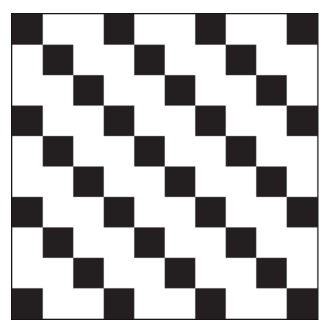
$\mathbf{(a + b + c)^4}$



$\mathbf{\partial((a + b + c)^4)/\partial a}$



$\mathbf{\partial^2((a + b + c)^4)/\partial a^2}$



$\mathbf{\partial^3((a + b + c)^4)/\partial a^3}$

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Appendix C — Integration

$(a + b + c)^4 = a^4 + 4a^3b + 4ac^3 + 6a^2b^2 + 12a^2bc + 6a^2c^2 + 4ab^3 + 12ab^2c + 12abc^2 + 4bc^3 + b^4 + 4b^3c + 6b^2c^2 + c^4$

aaaaaaabaaabaaabaaabaaacaaacaaacaaacaabbaabbaabbaabbaabbaabbaabcaabcaabcaabcaabcaabcaabcaabcaabcaabcaabcaa ccaaccaaccaaccaaccaaccabbbabbbabbbabbbabbcabbcabbcabbcabbcabbcabbcabbcabbcabbcabbcabccabccabccabccabccabcca bccabccabccabccabccabccacccacccacccacccbbbbbbbcbbbcbbbcbbbcbbccbbccbbccbbccbbccbbccbcccbcccbcccbcccccc

$\int(a + b + c)^4 da = 0.2a^5 + a^4b + a^4c + 2a^3\mathrm{b}2 + 4a^3bc + 2a^3c^2 + 2\mathrm{a}^2\mathrm{b}^3 + 6a^2b^2c + 6a^2bc^2 + 2a^2c^3$

aaaaabaaaacaaabbaaabbaaabcaaabcaaabcaaabcaaaccaaaccaabbbaabbbaabbcaabbcaabbcaabbcaabbcaabbcaabccaabccaabccaabc caabccaabccaacccaaccc

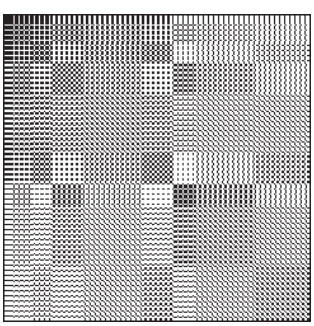
$\iint(a + b + c)^4 da^2 \approx 0.033a^6 + 0.2a^5b + 0.2a^5c + 0.5a^4b^2 + a^4bc + 0.5a^4c^2 + 0.667a^3b^3 + 2a^3b^2c + 2a^3bc^2 + 0.667a^3c^3$

aaaaaaaaabcaaaaaabaaabbcaaabbcaaabccaaabccaaac

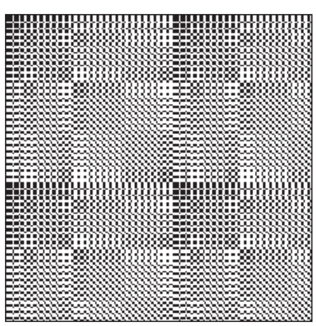
$\iiint(a + b + c)^4 da^3 \approx 0.005a^7 + 0.033a^6b + 0.033a^6c + 0.1a^5b^2 + 0.2a^5bc + 0.1a^5c^2 + 0.167a^4b^3 + 0.5a^4b^2c + 0.5a^4bc^2 + 0.167a^4c^3$

aaaaaaaaa

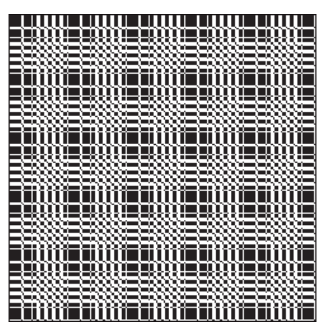
$\iiiint(a + b + c)^4 da^4 \approx 0.001a^8 + 0.005a^7\mathrm{b} + 0.005a^7\mathrm{c} + 0.017a^6\mathrm{b}^2 + 0.033a^6bc + 0.017a^6c^2 + 0.033a^5b^4 + 0.1a^5b^2c + 0.1a^5bc^2 + 0.033a^5c^3$



$\boldsymbol{(a + b + c)^4}$



$\boldsymbol{\int(a + b + c)^4 da}$



$\boldsymbol{\iint(a + b + c)^4 da^2}$

$\boldsymbol{\iiint(a + b + c)^4 da^3}$