Designing with Power Sequences

It may seem logical that sources of design inspiration among integer sequences should be sought among the complex, the unusual, and even the exotic. In fact, other articles in this series have explored these areas [1-2].

But relatively simple integer sequences contain hidden within them fascinating patterns that can be used to advantage in design, and in particular for threading and treadling sequences in weaving drafts.

Consider the squares:

1, 4, 9, 16, 25, …

There are many interesting number theoretic aspects to the squares, including the fact that all positive integers can be represented as the sum of four squares.

Nevertheless, the squares don't have any obvious attractive characteristics that might be useful in weave design. And since they increase in magnitude rapidly, they seem far from the realm of threading and treadling sequences — t-sequences.

It is necessary to "tame" such sequences to bring them into the domain of t-sequences — to reduce the large numbers to the realm of the shafts and treadles that are available on looms.

The natural way to do this, which preserves many of their intrinsic properties, is *modular reduction*, which is treated in detail in another article [3]. The basic idea is to take remainders on division by a number, called the *modulus*. These remainders are then adjusted to start at 1 rather than 0 to conform to the convention for numbering shafts and treadles. The result is a *residue sequence* that is suitable for threading and treadling.

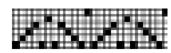
There is another problem to consider. Residue sequences often are missing values. For example, the residue sequence for the squares modulo 2 is

1, 4, 3, 4, 1, 6, … *repeated*

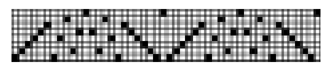
One way to handle this problem is to reassign the values to fill in the gaps. A systematic way has useful properties is called *normalization* [4]. Normalization reassigns successive value to the integers in order: 1, 2, 3, … . The normalized sequence for the example above is

1, 2, 3, 2, 1, 4, … *repeated*

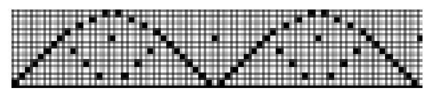
Here are three normalized residue sequences for the squares, with two repeats shown in graphical format:



squares mod 24



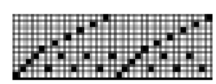
squares mod 48



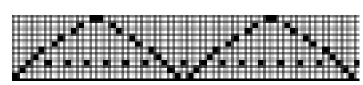
squares mod 64

Higher Powers

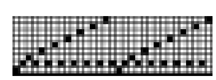
Here are some examples from higher powers, with two repeats shown in graphical format:



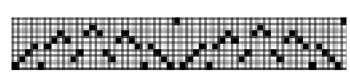
cubes mod 16



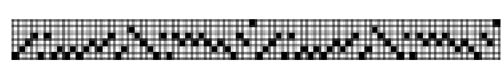
tenth powers mod 27



thirteenth powers mod 16



twentieth powers mod 26



twenty-seventh powers mod 38

Periods

All residue sequences derived from power sequences are periodic — they repeat. In many cases the periods are the same as the moduli. The exception, in which the periods are shorter, occur when the modulus is evenly divisible by the square of one of the factors of the power.

For example, for squares, moduli divisible by $2^2 = 4$ have residue sequences less than the moduli:

modulus	*period*
4	2
8	4
12	6
16	8
20	10
24	12
28	14
32	16
…	…

For cubes, moduli divisible by $3^2 = 9$ have periods less than the moduli:

modulus	*period*
9	3
18	6
27	9
…	…

Note that for primes, which have only 1 and themselves as factors, the number of sequences with periods less than the moduli is relatively small. For example, for power 5, only moduli divisible by $5^2 = 25$ have periods less than the moduli.

In the case of composite moduli (those with more than one divisor), the situation is more complicated with respect to the lengths of the periods.

For example, for power 4 = 2 × 2, the results are

modulus	*period*
4	2
8	2
12	6
16	2
20	10
24	6
28	8
32	8
…	…

One important consequence of shorter periods when the moduli are divisible by the square of a factor of the power is that there are sequences with relatively short periods even for large powers and large moduli.

Straight Draws

As mentioned in Reference 4, many normalized residue sequences are straight draws and uninteresting for design purposes.

The numbers of straight draws for powers from 2 to 16 and modulo 2 to 64 are:

power	*straight draws*
2	2
3	25
4	4
5	32
6	3
7	36
8	5
9	27
10	3
11	36
12	4
13	37
14	3
15	22
16	6

Note than even powers have higher percentages of residue sequences that are potentially interesting.

Number of Distinct Values

Except for straight draws, the number of distinct values in normalized residue sequences (and hence the number of shafts or treadles required) is always less that the modulus, and considerably so.

Most often, but not always, the number of distinct values is even. There is, however, no evident regularity in this regard.

In general, however, there are sequences with small numbers of distinct values even for large powers and large moduli. See Appendix A summarizes the properties of normalized residue sequences for powers from 2 to 32 and moduli from 2 to 64.

Distinct Sequences

The same sequence may be found in normalized residue sequences for different powers and different moduli.

For powers from 2 to 32 and moduli from 2 to 64, there are 31 × 63 = 1,891 sequences in all. Of these, 537 are straight draws. Among the remaining 1,351, there are only 207 distinct sequences.

Of these, 12 require more than 32 shafts or treadles and can be discarded for most looms. Of the 195 remaining, some are unsuitable for thread-by-thread drafting, although they may be candidates for profile drafting. Some of the remaining sequences are uninteresting or aesthetically flawed. But among these sequences are many interesting candidates for weave design. See Appendix B.

Comment

Power sequences provide one example of simple integer sequences that have the potential for the design of interesting weaves. But they are far from unique in this characteristic. There is a vast territory to explore.

References

1. *The Morse-Thue Sequence*, Ralph E. Griswold, 2004:
http://www.cs.arizona.edu/patterns/weaving/webdocs/gre_mt.pdf.

2. *Signature Sequences in Weave Design*, Ralph E. Griswold, 2004:
http://www.cs.arizona.edu/patterns/weaving/webdocs/gre_sig.pdf.

3. *Drafting with Sequences*, Ralph E. Griswold, 2004:
http://www.cs.arizona.edu/patterns/weaving/webdocs/gre_seqd.pdf.

4. *Normalized Sequences*, Ralph E. Griswold, 2004:
http://www.cs.arizona.edu/patterns/weaving/webdocs/gre_norm.pdf.

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Appendix A — Properties of Normalized Residue Sequences Derived from Power Sequences

2				3				4				5				6			
m	*s*	*p*	*n*	*m*	*s*	*p*	*n*	*m*	*s*	*p*	*n*	*m*	*s*	*p*	*n*	*m*	*s*	*p*	*n*
2	2	2	x	2	2	2	x	2	2	2	x	2	2	2	x	2	2	2	x
3	2	3		3	3	3	x	3	2	3		3	3	3	x	3	2	3	
4	2	2	x	4	3	4		4	2	2	x	4	3	4		4	2	2	x
5	3	5		5	5	5	x	5	2	5		5	5	5	x	5	3	5	
6	4	6		6	6	6	x	6	4	6		6	6	6	x	6	4	6	
7	4	7		7	3	7		7	4	7		7	7	7	x	7	2	7	
8	3	4	*	8	5	8		8	2	2	x	8	5	8		8	2	2	x
9	4	9		9	3	3	x	9	4	9		9	7	9		9	2	3	*
10	6	10		10	10	10	x	10	4	10		10	10	10	x	10	6	10	
11	6	11		11	11	11	x	11	6	11		11	3	11		11	6	11	
12	4	6	*	12	9	12		12	4	6	*	12	9	12		12	4	6	*
13	7	13		13	5	13		13	4	13		13	13	13	x	13	3	13	
14	8	14		14	6	14		14	8	14		14	14	14	x	14	4	14	
15	6	15		15	15	15	x	15	4	15		15	15	15	x	15	6	15	
16	4	8	*	16	10	16		16	2	2	x	16	9	16		16	3	8	*
17	9	17		17	17	17	x	17	5	17		17	17	17	x	17	9	17	
18	8	18		18	6	6	x	18	8	18		18	14	18		18	4	6	*
19	10	19		19	7	19		19	10	19		19	19	19	x	19	4	19	
20	6	10	*	20	15	20		20	4	10	*	20	15	20		20	6	10	*
21	8	21		21	9	21		21	8	21		21	21	21	x	21	4	21	
22	12	22		22	22	22	x	22	12	22		22	6	22		22	12	22	
23	12	23		23	23	23	x	23	12	23		23	23	23	x	23	12	23	
24	6	12	*	24	15	24		24	4	6	*	24	15	24		24	4	6	*
25	11	25		25	21	25		25	6	25		25	5	5	x	25	11	25	
26	14	26		26	10	26		26	8	26		26	26	26	x	26	6	26	
27	11	27		27	7	9	*	27	10	27		27	19	27		27	4	9	*
28	8	14	*	28	9	28		28	8	14	*	28	21	28		28	4	14	*
29	15	29		29	29	29	x	29	8	29		29	29	29	x	29	15	29	
30	12	30		30	30	30	x	30	8	30		30	30	30	x	30	12	30	

Legend:

Powers are listed along the top row.
m = modulus
s = distinct values (shafts)
p = period
n = notes

Notes:
x = straight draw
* = period less than modulus

2				3				4				5				6			
m	*s*	*p*	*n*	*m*	*s*	*p*	*n*	*m*	*s*	*p*	*n*	*m*	*s*	*p*	*n*	*m*	*s*	*p*	*n*
31	16	31		31	11	31		31	16	31		31	7	31		31	6	31	
32	7	16	*	32	19	32		32	4	8	*	32	17	32		32	5	16	*
33	12	33		33	33	33	x	33	12	33		33	9	33		33	12	33	
34	18	34		34	34	34	x	34	10	34		34	34	34	x	34	18	34	
35	12	35		35	15	35		35	8	35		35	35	35	x	35	6	35	
36	8	18	*	36	9	12	*	36	8	18	*	36	21	36		36	4	6	*
37	19	37		37	13	37		37	10	37		37	37	37	x	37	7	37	
38	20	38		38	14	38		38	20	38		38	38	38	x	38	8	38	
39	14	39		39	15	39		39	8	39		39	39	39	x	39	6	39	
40	9	20	*	40	25	40		40	4	10	*	40	25	40		40	6	10	*
41	21	41		41	41	41	x	41	11	41		41	9	41		41	21	41	
42	16	42		42	18	42		42	16	42		42	42	42	x	42	8	42	
43	22	43		43	15	43		43	22	43		43	43	43	x	43	8	43	
44	12	22	*	44	33	44		44	12	22	*	44	9	44		44	12	22	*
45	12	45		45	15	15	x	45	8	45		45	35	45		45	6	15	*
46	24	46		46	46	46	x	46	24	46		46	46	46	x	46	24	46	
47	24	47		47	47	47	x	47	24	47		47	47	47	x	47	24	47	
48	8	24	*	48	30	48		48	4	6	*	48	27	48		48	6	24	*
49	22	49		49	15	49		49	22	49		49	43	49		49	8	49	
50	22	50		50	42	50		50	12	50		50	10	10	x	50	22	50	
51	18	51		51	51	51	x	51	10	51		51	51	51	x	51	18	51	
52	14	26	*	52	15	52		52	8	26	*	52	39	52		52	6	26	*
53	27	53		53	53	53	x	53	14	53		53	53	53	x	53	27	53	
54	22	54		54	14	18	*	54	20	54		54	38	54		54	8	18	*
55	18	55		55	55	55	x	55	12	55		55	15	55		55	18	55	
56	12	28	*	56	15	56		56	8	14	*	56	35	56		56	4	14	*
57	20	57		57	21	57		57	20	57		57	57	57	x	57	8	57	
58	30	58		58	58	58	x	58	16	58		58	58	58	x	58	30	58	
59	30	59		59	59	59	x	59	30	59		59	59	59	x	59	30	59	
60	12	30	*	60	45	60		60	8	30	*	60	45	60		60	12	30	*
61	31	61		61	21	61		61	16	61		61	13	61		61	11	61	
62	32	62		62	22	62		62	32	62		62	14	62		62	12	62	
63	16	63		63	9	21	*	63	16	63		63	49	63		63	4	21	*
64	12	32	*	64	37	64		64	6	16	*	64	34	64		64	9	32	*

7				8				9				10				11			
m	*s*	*p*	*n*	*m*	*s*	*p*	*n*	*m*	*s*	*p*	*n*	*m*	*s*	*p*	*n*	*m*	*s*	*p*	*n*
2	2	2	x	2	2	2	x	2	2	2	x	2	2	2	x	2	2	2	x
3	3	3	x	3	2	3		3	3	3	x	3	2	3		3	3	3	x
4	3	4		4	2	2	x	4	3	4		4	2	2	x	4	3	4	
5	5	5	x	5	2	5		5	5	5	x	5	3	5		5	5	5	x
6	6	6	x	6	4	6		6	6	6	x	6	4	6		6	6	6	x
7	7	7	x	7	4	7		7	3	7		7	4	7		7	7	7	x
8	5	8		8	2	2	x	8	5	8		8	2	2	x	8	5	8	
9	7	9		9	4	9		9	3	3	x	9	4	9		9	7	9	
10	10	10	x	10	4	10		10	10	10	x	10	6	10		10	10	10	x
11	11	11	x	11	6	11		11	11	11	x	11	2	11		11	11	11	x
12	9	12		12	4	6	*	12	9	12		12	4	6	*	12	9	12	
13	13	13	x	13	4	13		13	5	13		13	7	13		13	13	13	x
14	14	14	x	14	8	14		14	6	14		14	8	14		14	14	14	x
15	15	15	x	15	4	15		15	15	15	x	15	6	15		15	15	15	x
16	9	16		16	2	2	x	16	9	16		16	3	8	*	16	9	16	
17	17	17	x	17	3	17		17	17	17	x	17	9	17		17	17	17	x
18	14	18		18	8	18		18	6	6	x	18	8	18		18	14	18	
19	19	19	x	19	10	19		19	3	19		19	10	19		19	19	19	x
20	15	20		20	4	10	*	20	15	20		20	6	10	*	20	15	20	
21	21	21	x	21	8	21		21	9	21		21	8	21		21	21	21	x
22	22	22	x	22	12	22		22	22	22	x	22	4	22		22	22	22	x
23	23	23	x	23	12	23		23	23	23	x	23	12	23		23	3	23	
24	15	24		24	4	6	*	24	15	24		24	4	6	*	24	15	24	
25	21	25		25	6	25		25	21	25		25	3	5	*	25	21	25	
26	26	26	x	26	8	26		26	10	26		26	14	26		26	26	26	x
27	19	27		27	10	27		27	3	3	x	27	10	27		27	19	27	
28	21	28		28	8	14	*	28	9	28		28	8	14	*	28	21	28	
29	5	29		29	8	29		29	29	29	x	29	15	29		29	29	29	x
30	30	30	x	30	8	30		30	30	30	x	30	12	30		30	30	30	x
31	31	31	x	31	16	31		31	11	31		31	4	31		31	31	31	x
32	17	32		32	2	2	x	32	17	32		32	5	16	*	32	17	32	

7				8				9				10				11			
m	*s*	*p*	*n*	*m*	*s*	*p*	*n*	*m*	*s*	*p*	*n*	*m*	*s*	*p*	*n*	*m*	*s*	*p*	*n*
33	33	33	×	33	12	33		33	33	33	×	33	4	33		33	33	33	×
34	34	34	×	34	6	34		34	34	34	×	34	18	34		34	34	34	×
35	35	35	×	35	8	35		35	15	35		35	12	35		35	35	35	×
36	21	36		36	8	18	*	36	9	12	*	36	8	18	*	36	21	36	
37	37	37	×	37	10	37		37	5	37		37	19	37		37	37	37	×
38	38	38	×	38	20	38		38	6	38		38	20	38		38	38	38	×
39	39	39	×	39	8	39		39	15	39		39	14	39		39	39	39	×
40	25	40		40	4	10	*	40	25	40		40	6	10	*	40	25	40	
41	41	41	×	41	6	41		41	41	41	×	41	5	41		41	41	41	×
42	42	42	×	42	16	42		42	18	42		42	16	42		42	42	42	×
43	7	43		43	22	43		43	15	43		43	22	43		43	43	43	×
44	33	44		44	12	22	*	44	33	44		44	4	22	*	44	33	44	
45	35	45		45	8	45		45	15	15	×	45	12	45		45	35	45	
46	46	46	×	46	24	46		46	46	46	×	46	24	46		46	6	46	
47	47	47	×	47	24	47		47	47	47	×	47	24	47		47	47	47	×
48	27	48		48	4	6	*	48	27	48		48	6	24	*	48	27	48	
49	7	7	×	49	22	49		49	15	49		49	22	49		49	43	49	
50	42	50		50	12	50		50	42	50		50	6	10	*	50	42	50	
51	51	51	×	51	6	51		51	51	51	×	51	18	51		51	51	51	×
52	39	52		52	8	26	*	52	15	52		52	14	26	*	52	39	52	
53	53	53	×	53	14	53		53	53	53	×	53	27	53		53	53	53	×
54	38	54		54	20	54		54	6	6	×	54	20	54		54	38	54	
55	55	55	×	55	12	55		55	55	55	×	55	6	55		55	55	55	×
56	35	56		56	8	14	*	56	15	56		56	8	14	*	56	35	56	
57	57	57	×	57	20	57		57	9	57		57	20	57		57	57	57	×
58	10	58		58	16	58		58	58	58	×	58	30	58		58	58	58	×
59	59	59	×	59	30	59		59	59	59	×	59	30	59		59	59	59	×
60	45	60		60	8	30	*	60	45	60		60	12	30	*	60	45	60	
61	61	61	×	61	16	61		61	21	61		61	7	61		61	61	61	×
62	62	62	×	62	32	62		62	22	62		62	8	62		62	62	62	×
63	49	63		63	16	63		63	9	21	*	63	16	63		63	49	63	
64	33	64		64	3	8	*	64	33	64		64	9	32	*	64	33	64	

12				13				14				15				16			
m	*s*	*p*	*n*	*m*	*s*	*p*	*n*	*m*	*s*	*p*	*n*	*m*	*s*	*p*	*n*	*m*	*s*	*p*	*n*
2	2	2	x	2	2	2	x	2	2	2	x	2	2	2	x	2	2	2	x
3	2	3		3	3	3	x	3	2	3		3	3	3	x	3	2	3	
4	2	2	x	4	3	4		4	2	2	x	4	3	4		4	2	2	x
5	2	5		5	5	5	x	5	3	5		5	5	5	x	5	2	5	
6	4	6		6	6	6	x	6	4	6		6	6	6	x	6	4	6	
7	2	7		7	7	7	x	7	4	7		7	3	7		7	4	7	
8	2	2	x	8	5	8		8	2	2	x	8	5	8		8	2	2	x
9	2	3	*	9	7	9		9	4	9		9	3	3	x	9	4	9	
10	4	10		10	10	10	x	10	6	10		10	10	10	x	10	4	10	
11	6	11		11	11	11	x	11	6	11		11	3	11		11	6	11	
12	4	6	*	12	9	12		12	4	6	*	12	9	12		12	4	6	*
13	2	13		13	13	13	x	13	7	13		13	5	13		13	4	13	
14	4	14		14	14	14	x	14	8	14		14	6	14		14	8	14	
15	4	15		15	15	15	x	15	6	15		15	15	15	x	15	4	15	
16	2	2	x	16	9	16		16	3	8	*	16	9	16		16	2	2	x
17	5	17		17	17	17	x	17	9	17		17	17	17	x	17	2	17	
18	4	6	*	18	14	18		18	8	18		18	6	6	x	18	8	18	
19	4	19		19	19	19	x	19	10	19		19	7	19		19	10	19	
20	4	10	*	20	15	20		20	6	10	*	20	15	20		20	4	10	*
21	4	21		21	21	21	x	21	8	21		21	9	21		21	8	21	
22	12	22		22	22	22	x	22	12	22		22	6	22		22	12	22	
23	12	23		23	23	23	x	23	12	23		23	23	23	x	23	12	23	
24	4	6	*	24	15	24		24	4	6	*	24	15	24		24	4	6	*
25	6	25		25	21	25		25	11	25		25	5	5	x	25	6	25	
26	4	26		26	26	26	x	26	14	26		26	10	26		26	8	26	
27	4	9	*	27	19	27		27	10	27		27	7	9	*	27	10	27	
28	4	14	*	28	21	28		28	8	14	*	28	9	28		28	8	14	*
29	8	29		29	29	29	x	29	3	29		29	29	29	x	29	8	29	
30	8	30		30	30	30	x	30	12	30		30	30	30	x	30	8	30	
31	6	31		31	31	31	x	31	16	31		31	3	31		31	16	31	
32	3	8	*	32	17	32		32	5	16	*	32	17	32		32	2	2	x

12				13				14				15				16			
m	*s*	*p*	*n*	*m*	*s*	*p*	*n*	*m*	*s*	*p*	*n*	*m*	*s*	*p*	*n*	*m*	*s*	*p*	*n*
33	12	33		33	33	33	x	33	12	33		33	9	33		33	12	33	
34	10	34		34	34	34	x	34	18	34		34	34	34	x	34	4	34	
35	4	35		35	35	35	x	35	12	35		35	15	35		35	8	35	
36	4	6	*	36	21	36		36	8	18	*	36	9	12	*	36	8	18	*
37	4	37		37	37	37	x	37	19	37		37	13	37		37	10	37	
38	8	38		38	38	38	x	38	20	38		38	14	38		38	20	38	
39	4	39		39	39	39	x	39	14	39		39	15	39		39	8	39	
40	4	10	*	40	25	40		40	6	10	*	40	25	40		40	4	10	*
41	11	41		41	41	41	x	41	21	41		41	9	41		41	6	41	
42	8	42		42	42	42	x	42	16	42		42	18	42		42	16	42	
43	8	43		43	43	43	x	43	4	43		43	15	43		43	22	43	
44	12	22	*	44	33	44		44	12	22	*	44	9	44		44	12	22	*
45	4	15	*	45	35	45		45	12	45		45	15	15	x	45	8	45	
46	24	46		46	46	46	x	46	24	46		46	46	46	x	46	24	46	
47	24	47		47	47	47	x	47	24	47		47	47	47	x	47	24	47	
48	4	6	*	48	27	48		48	6	24	*	48	27	48		48	4	6	*
49	8	49		49	43	49		49	4	7	*	49	15	49		49	22	49	
50	12	50		50	42	50		50	22	50		50	10	10	x	50	12	50	
51	10	51		51	51	51	x	51	18	51		51	51	51	x	51	4	51	
52	4	26	*	52	39	52		52	14	26	*	52	15	52		52	8	26	*
53	14	53		53	5	53		53	27	53		53	53	53	x	53	14	53	
54	8	18	*	54	38	54		54	20	54		54	14	18	*	54	20	54	
55	12	55		55	55	55	x	55	18	55		55	15	55		55	12	55	
56	4	14	*	56	35	56		56	8	14	*	56	15	56		56	8	14	*
57	8	57		57	57	57	x	57	20	57		57	21	57		57	20	57	
58	16	58		58	58	58	x	58	6	58		58	58	58	x	58	16	58	
59	30	59		59	59	59	x	59	30	59		59	59	59	x	59	30	59	
60	8	30	*	60	45	60		60	12	30	*	60	45	60		60	8	30	*
61	6	61		61	61	61	x	61	31	61		61	5	61		61	16	61	
62	12	62		62	62	62	x	62	32	62		62	6	62		62	32	62	
63	4	21	*	63	49	63		63	16	63		63	9	21	*	63	16	63	
64	5	16	*	64	33	64		64	9	32	*	64	33	64		64	2	2	x

Appendix B -- Normalized Residue Sequences Derived from Power Sequences