Recurrence Relations

A recurrence relation gives the terms of a sequence as a function of previous terms. For example, the Fibonacci sequence is given by the recurrence

$$a_n = a_{n-1} + a_{n-2}$$

with the initial terms $a_1 = a_2 = 1$ to get the sequence started. Different initial terms produce different but related sequences.

The number of initial terms required is determined by how far back in the sequence terms are specified — called the *order* of the recurrence relation. For example,

$$a_n = a_{n-1} + 2a_{n-3}$$

is a recurrence relation of order 3 and requires three initial terms, a_1, a_2, and a_3, to specify the sequence it produces.

The examples given above are linear recurrence relations with constant coefficients — LRRCs for short — and are instances of the general form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} \qquad (1)$$

where only the first powers of previous terms are used and the coefficients are constant.

There are other kinds of recurrence relations. For example,

$$a_n = a^2_{n-1} + a^2_{n-2} + a_{n-4}$$

is a quadratic recurrence of order 4, while

$$a_n = a_{n-1} + na_{n-2}$$

is a linear recurrence of order 2 but with a nonconstant coefficient.

LRRCs are important in subjects including pseudo-random number generation, circuit design, and cryptography, and they have been studied extensively. LRRCs also have periodic residue sequences [1], which is the main reason for our interest in them. Despite the importance of LRRCs and the work done on them, much about them remains unknown. Very little of a general nature is known about nonlinear recurrence relations. We'll focus mainly on LRRCs.

LRRCs

LRRC Canonical Form

Equation 1 above shows the canonical form for LRRCs. This form does not provide for a constant term, as in

$$a_n = a_{n-1} + 1$$

The reason for not having a constant term in the canonical form has to do with manipulations of LRRCs in which a constant term would require special handling.

A linear recurrence of order k with a constant term can be converted to a linear recurrence of order $k + 1$ in canonical form. Consider the example above:

$$a_n = a_{n-1} + 1 \qquad (2)$$

From this it follows that

$$a_{n-1} = a_{n-2} + 1 \qquad (3)$$

Subtracting Equation 3 from Equation 2, we get

$$a_n - a_{n-1} = a_{n-1} + 1 - a_{n-2} - 1$$

and hence

$$a_n = 2a_{n-1} - a_{n-2}$$

which is in the required canonical form.

Problems Related to LRRCs

There are many interesting problems related to LRRCs. In the article on residue sequences, we touched on the properties of their residue sequences. Other problems of interest are:

- computing the sequence for an LRRC
- determining if a sequence can be represented by an LRRC and, if so, finding it
- solving an LRRC to produce an explicit formula for its nth term

An LRRC Generator

An LRRC can be completely characterized by two lists: one containing its coefficients and another containing its initial terms. For an LRRC of order k, both lists are of length k. For example, the recurrence relation

$$a_n = a_{n-1} + 2a_{n-3}$$

has the coefficient list [1, 0, 2]; the initials list, as always, determines the actual sequence. For example, the initials list [1,1,0] produces the sequence

1, 1, 0, 2, 4, 4, 8, 16, 24, 40, 72, 120, …

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Finding LRRCs

Many sequences can be represented by LRRCs, even if the recurrences are not obvious.

The *difference method* often works and it can be done by hand or with a simple program [2]. This method starts with a row containing the terms of the original sequence. The second row consists of the differences of successive terms in the first row, and so on. The rows are labeled Δ^0, Δ^1, Δ^2, … . Here's an example:

Δ^0	1	7	18	34	55	81	112	148	189	…
Δ^1	6	11	16	21	26	31	36	41	…	
Δ^2	5	5	5	5	5	5	5	…		
Δ^3	0	0	0	0	0	0	…			

If a constant row appears, as it does in this example, the process is complete, there is an LRRC, and it can be obtained by using Equation 4 below, which is a consequence of the way the differences are computed:

$$\Delta^k a_n = \sum_{i=0}^{k} (-1)^i \binom{k}{i} a_{n+k-i} \qquad (4)$$

where i is the binomial coefficient

$$\binom{k}{i} = \frac{k!}{(k-i)!\,i!}$$

To get an LRRC in canonical form, it is necessary to go to a row of zeroes; Δ^3 in this case. Therefore, by Equation 4

$$\Delta^3 a_n = \sum_{i=0}^{3} (-1)^i \binom{3}{i} a_{n+3-i} = 0$$

Expanding this, we get

$$\binom{3}{0} a_{n+3} - \binom{3}{1} a_{n+2} + \binom{3}{2} a_{n+1} - \binom{3}{3} a_n = 0$$

and hence

$$a_{n+3} - 3a_{n+2} + 3a_{n+1} - a_n = 0$$

from which we get the LRRC

$$a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$$

The initial terms are, of course, the first three in Δ^0.

Any recurrence derived from a finite number of terms is, of course, conjectural.

Explicit Formulas for LRRC Terms

Any sequence that leads to a 0 Δ sequence can be represented by a polynomial in n. Conversely, all polynomials in n can be represented by a single LRRC; the coefficients of the polynomial only affect the initial terms for the LRRC.

This follows from another equation that results from the method of differences:

$$a_{n+m} = \sum_{k=0}^{n} \binom{n}{k} \Delta^k a_m \qquad (5)$$

From this, we can obtain an explicit formula for the nth term of the corresponding LRRC. Setting m to 1 in Equation 5 gives

$$a_{n+1} = 1\binom{n}{0} + 6\binom{n}{1} + 5\binom{n}{2} + 0\binom{n}{3}$$

(1, 6, 5, and 0 are the leading terms in Δ^0, Δ^1, Δ^2, and Δ^3.) This evaluates to

$$a_{n+1} = 1 + \frac{7}{2}n + \frac{5}{2}n^2$$

References

1. *Residue Sequences in Weave Design*, Ralph E. Griswold, 2000: //http:www.cs.arizona.edu/patterns/weaving/webdocs/gre_res.pdf.

2. *The Encyclopedia of Integer Sequences*, N. J. A. Sloane and Simon Plouffe, Academic Press, 1995, pp. 10-13.

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