Pattern Extension Schemata, Part 2: Symbols

Mathematicians love to invent notation, the more obscure and arcane, the better. Computer scientists love to play at being mathematicians.

— anonymous

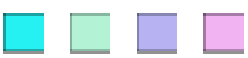
In the first article in this series [1], we described the inversion schema for extending a pattern by appending to it copies of its face and back.

In this article, we'll introduce some notation to describe geometrical transformations on patterns and schemata for pattern extension. This notation used color and pictograms to make the transformations easy to visualize.

This notation lays the groundwork for articles to follow.

Pattern Symbols

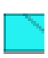
Patterns are indicated by colored squares. This is an abstraction. The actual patterns need not be square and color serves only to identify different patterns. (If you are a programmer, think of a color as corresponding to a variable.) Here some pattern symbols:



Transformation Symbols

Geometric transformations are indicated by markings inside pattern squares. The pictograms for the basic geometrical transformations follow.

There are three rotations by multiples of 90°:

 rotation 90° clockwise

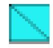
 rotation 180°

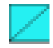
 rotation 90° counterclockwise

There are four flips, two around vertical and horizontal centers and two around the diagonals (diagonals are designated by the upper corner they touch):

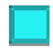
 horizontal flip (around vertical axis)

 vertical flip (around horizontal axis)

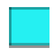
 left diagonal flip

 right diagonal flip

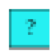
Finally, there is inversion — changing black pattern cells to white and vice versa — geometrically flipping the pattern from front to back:

 inversion

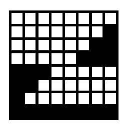
For completeness, we need the identity transformation, which does nothing:

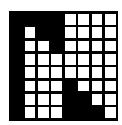
 identity

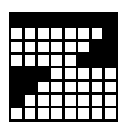
And for unspecified transformations, we'll use the following pictogram:

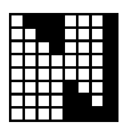
 unspecified transformation

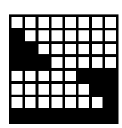
Here are examples of the transformations applied to an actual pattern as given by the identity transformation:

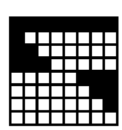
 identity

 rotation 90° clockwise

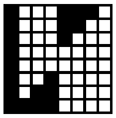
 rotation 180°

 rotation 90° counterclockwise

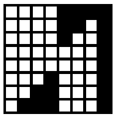
 horizontal flip

 vertical flip

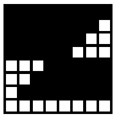
November 28, 2001; last revised August 1, 2004



left diagonal flip

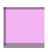


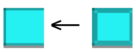
right diagonal flip



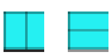
inversion

We'll use a left-going arrow to indicate a pattern produced by a transformation of another, as in

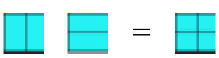
in which  is the inversion of . When a pattern is itself changed by transformation, the result of the transformed pattern replaces it, as in



There are nine basic transformations including the identity transformation. Other transformations can be obtained by applying basic transformations in succession, such as a horizontal flip followed by a vertical flip, which we indicate by



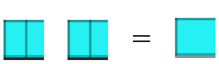
We could combine these in a single pictogram:



This is unambiguous, since horizontal and vertical flips are commutative; that is, they produce the same result regardless of the order in which they are applied.

Such pictogram combination is economical of space, but carried to extremes leads to pictograms that are confusing — the exact opposite of what we are trying to accomplish by using pictograms.

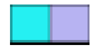
There are various relationships between the basic transformations. For example, horizontal flips are reflective — a horizontal flip is its own inverse: Applying two horizontal flips produces the identity pattern:



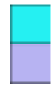
We won't go into the various relationships among transformations here. They can be found in books on group theory, symmetry, and tiling [2,3].

Concatenation

There are two basic ways that patterns can be concatenated (appended) to form larger patterns:

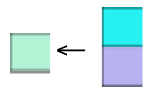


horizontal concatenation



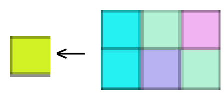
vertical concatenation

The result of concatenation is, of course, a pattern, as indicated by



There are restrictions on concatenation: it is only allowed if the abutting edges are of the same length so that a rectangle is formed

Concatenations can be combined as needed to form larger patterns, as in



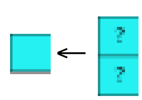
Single-Pattern Schemata

The simplest pattern extension schemata use only a single pattern, concatenating various transformations of it. The result then replaces the original pattern.

Schemata can be classified according to the concatenations involved. The three simplest are vertical concatenation (1 × 2 schemata), horizontal concatenation (2 × 1 schemata), and one of these followed by the other (2 × 2 schemata).

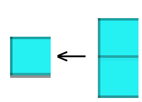
The general forms of the schemata are given by templates in which the operations on component patterns are unspecified.

1 × 2 Schemata

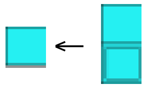


1 × 2 schema template

The 1 × 2 repetition and inversion schemata are:

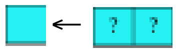


1 × 2 repetition schema



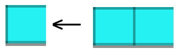
1 × 2 inversion schema

2 × 1 Schemata

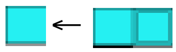


2 × 1 schema template

The 2 × 1 repetition and inversion schemata are:



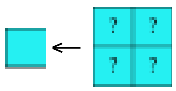
2 × 1 repetition schema



2 × 1 inversion schema

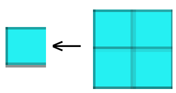
There are nine basic transformations, so there are $9^2 = 81$ possible 1 × 2 and 2 × 1 basic schemata If operations are compounded, there are, of course, many more.

2 × 2 Schemata

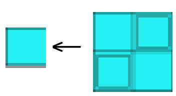


2 × 2 schema template

The 2 × 2 repetition and inversion schemata, which were described in the first article in this series, are:



2 × 2 repetition schema

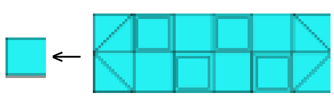


2 × 2 inversion schema

There are $9^4 = 6{,}561$ possible 2 × 2 basic schemata.

Larger Schemata

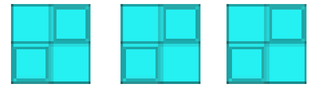
Larger single-pattern schemata are, of course, possible, as in



Large schemata have limited usefulness, since pattern size increases rapidly.

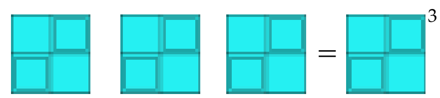
Schemata Chains

Chains of schemata can be use to repeat a schema or apply different schemata in succession, as in

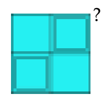


in which the inversion schemata is applied three times.

We can simplify such common cases by indicating the number of repetitions by an exponent, as in

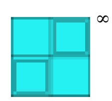


We'll use a question mark as an exponent to indicate an unspecified number of schema repetitions:



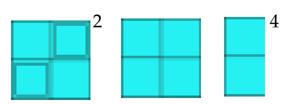
The number of schema repetitions could by 0 (no application), 1 (one application), 3 (three applications as in the example above), or any nonnegative number of applications.

On a more fanciful note, we can use the infinity symbol to indicate an endless number of applications. For example,



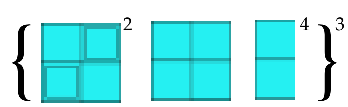
stands for an infinite fractal carpet. No such thing can actually exist, but such concepts have uses.

Different schemata can be applied in succession, as in



which stands for two applications of the 2 × 2 inversion schema, followed by one application of the 2 × 2 repetition schema, followed by four applications of the 1 × 2 repetition schemata.

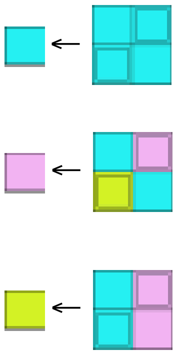
The repetition of a series of schemata is indicated by surrounding them with braces and adding an exponent, as in

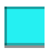
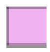


which stands for three applications of the schemata chain above.

Multiple-Pattern Schemata

Pattern extension schemata need not be limited to a single pattern; several patterns can be used as in this triple of three-pattern schemata:

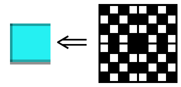


Here the three schemata are applied separately for each iteration,  ,  , and  all changing in parallel.

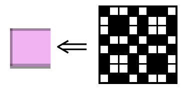
As with all such things, the possibilities are endless. The problem is sorting out interesting and useful instances. We won't pursue that here but just mention the possibility of metaschemata — higher-level schemata that describe pattern extension schemata.

Pattern Instantiation

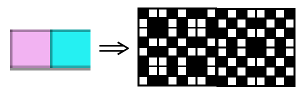
To complete our notation, we need a way of instantiating our abstract patterns with real patterns. To "load" a real pattern into an abstract pattern, we'll use a left-going double arrow, as in



and



To extract the real pattern associated with an abstract pattern, we'll use a double right-going arrow, as in



More to Come

That wraps up notation, at least for now.

In the next article on pattern extension schemata, we'll explore some specific schemata that can be used to produce useful and attractive results.

References

1. Griswold, Ralph E. "Pattern Extension Schemata, Part 1", 2004: http://www.cs.arizona.edu/patterns/weaving/webdocs/gre_sch1.pdf

2. Budden, F. J. *The Fascination of Groups*, Cambridge University Press, 1972.

3. Grünbaum, Branko and G. C. Shephard, *Tilings and Patterns*, W. H. Freeman, 1987.

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