Signature Sequences in Weave Design

Fractal Sequences

The term *fractal* is used in a variety of ways, formally and informally. It generally is understood that a fractal exhibits self similarity — that it appears the same at any scale.

This concept can be applied to integer sequences with respect to the magnitude and position of terms, various patterns, and so forth.

For example, Hofstadter's chaotic sequence [1], which is produced by the nested recurrence

$$q(i) = 1 \qquad i = 1, 2$$
$$q(i) = q(i - q(i-1)) + q(i - q(i-2)) \qquad i > 2$$

shows self similarity as can be seen in Figure 1.

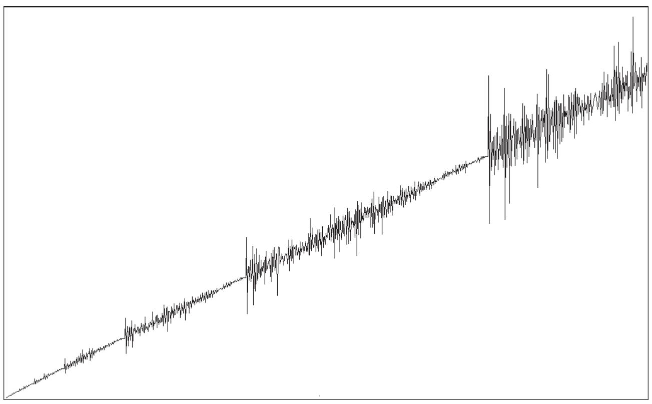

Figure 1. Hofstadter's Chaotic Sequence

In this sequence, sections of wide variations tailing off to minor variations double in length. The magnitude of the variations roughly doubles also. Within a section, you can see articulation of the preceeding section. Despite its tantalizing structure, this sequence is not strictly fractal.

Kimberling Fractal Sequences

Clark Kimberling has a specific definition of what he considers to be fractal integer sequences [2]. First, a Kimberling fractal sequence must be *infinitive*, which means that every positive integer occurs in it an infinite number of times.

An infinitive sequence $\{x_n\}$ has an *associative array*, $a(i,j)$, whose values are the jth indices for which $x_n = i$ for $i, j = 1, 2, 3, \ldots$.

$\{x_n\}$ is a fractal sequence if the following two conditions hold:

1. If $x_n = i+1$, then there is an $m < n$ such that $x_m = i$.

2. If $h < i$, then for every j there is exactly one k such that $a(i,j) < a(h,k) < a(i,j+1)$.

It seems to me there must be a simpler (or at least clearer) way to state this, but I haven't puzzled it out.

Such sequences have the property that if you strike out the first instance of every value, the resulting sequence is the same as the original (such sequences are, of course, infinite, which allows the concept of "same as" after deleting terms). An example of such a sequence is

1, 1, 1, 1, 2, 1, 2, 1, 3, 2, 1, 3, 2, 1, 3, …

Striking out the first instance of every term,

~~1~~, 1, 1, 1, ~~2~~, 1, 2, 1, ~~3~~, 2, 1, 3, 2, 1, 3, …

produces

1, 1, 1, 1, 2, 1, 2, 1, 3, 2, 1, 3, …

which is the same as the original sequence, as far as it goes.

There are two operations that when applied to fractal sequences yield fractal sequences: upper and lower trimming. Upper trimming is the "strike out" operation illustrated above. Lower trimming consists of subtracting 1 from every term and discarding 0s. For sequence above, the result of lower trimming is

1, 1, 2, 1, 2, 2, …

This is not the same as the original sequence, but it is a fractal sequence nonetheless.

Signature Sequences

An interesting class of Kimberling fractal sequences consists of *signature sequences* for irrational numbers. The signature sequence of the irrational number x is obtained by putting the numbers

$$i + j \times x \qquad i, j = 1, 2, 3, \ldots$$

in increasing order. Then the values of i for these numbers is the signature sequence for x, which I'll denote by $\mathbb{S}(x)$.

Here's the signature sequence for ϕ, the golden mean:

1, 2, 1, 3, 2, 4, 1, 3, 5, 2, 4, 1, 6, 3, 5, 2, 7, 4, 1, 6, 3, 8, 5, 2, 7, 4, 9, 1, 6, 3, 8, 5, 10, 2, 7, 4, 9, 1, 6, 11, …

Both upper trimming and lower trimming of a signature sequence leave the sequence unchanged.

Signature sequences have a characteristic appearance, but they vary considerably in detail de-

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pending of the value of x.

Signature sequences start with a run 1, 2, … $n+1$, where $n = \lfloor x \rfloor$, the integer part of x. The larger the value of x, the more quickly terms in the sequence get larger. Most signature sequences display runs, either upward or downward or both — which one is usually a matter of visual interpretation. At some point, most signature sequences become interleaved runs. This sometimes gives the illusion of curves.

Although signature sequences are defined only for irrational numbers, the algorithm works just as well for rational numbers. Although signature sequences for rational numbers are not fractal sequences, they are as close as you could determine manually. The structure of a signature sequence depends on the magnitude of x. Furthermore, there are irrational numbers arbitrarily close to any rational number. There is no difference in the initial terms of signature sequences for numbers that are close together. For example, $\mathbb{S}(3.0)$ and $\mathbb{S}(\pi)$ do not differ until their 117th terms.

It's also worth noting that there really is no way, in general, to perform exact computations for irrational numbers. Computers approximate real numbers, and hence irrational numbers, using floating-point arithmetic. A floating-point number representing an irrational number is just a (very good) rational approximation to the irrational number. For example, the standard 64-bit floating-point encoding for π is

$$7074237752028440/2^{51}$$

Figure 2 shows grid plots for some signature sequences. I didn't include signature sequences for large numbers because they become unwieldy.

Using Signature Sequences in Weaving Drafts

Signature sequences can be used as the basis for threading and treadling sequences. To use signature sequences for this purpose, it is necessary to bring the values of terms within the bounds of the number of shafts and treadles used. The mathematically reasonable way is to take their residues, modulo the number of shafts or treadles, using 1-based arithmetic [3]. Figure 3 shows residue sequences derived from signature sequences. In most cases, taking residues preserves the essential characteristics of signature sequences.

Sequences like these, if used directly, produce drawdown patterns that lack repeats or symmetry. More attractive patterns can be obtained by taking a small portion of a signature sequence and then reflecting it to get symmetric repeats.

Figure 4 shows a draft for such a sequence with 16 shafts and treadles and a $\frac{2}{\;\;2}$ twill tie-up.

It seems natural to use initial terms of a signature sequence. The structure of signature sequences, however, changes as the sequence goes on. Figure 5 shows magnified portions of the drawdown pattern for a signature sequence. This suggests that it might be worth trying subsequences of signature sequences in various locations.

Figure 6 shows some drawdown patterns for various combinations of signature sequences. All have 16 shafts and treadles and $\frac{2}{\;\;2}$ twill tie-ups.

References

1. *Gödel, Escher, Bach: An Eternal Golden Braid*, Douglas R. Hofstadter, Basic Books, 1979, pp. 137-138.

2. *CRC Concise Encyclopedia of Mathematics*, Eric W. Weisstein, Chapman & Hall/CRC, 1999, pp. 674.

3. Residue Sequences in Weave Design, Ralph E. Griswold, 2000:
http://www.cs.arizona.edu/patterns/weaving/gre_res.pdf

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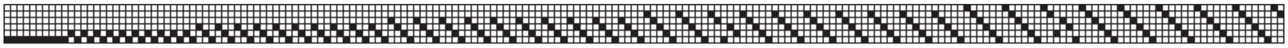
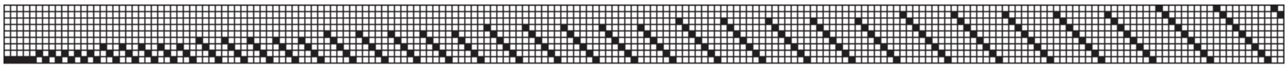
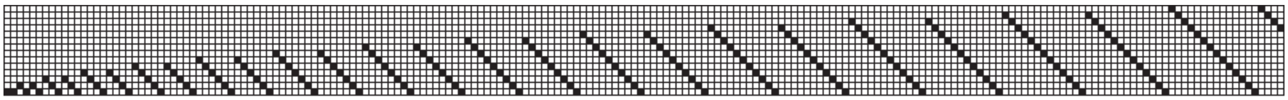


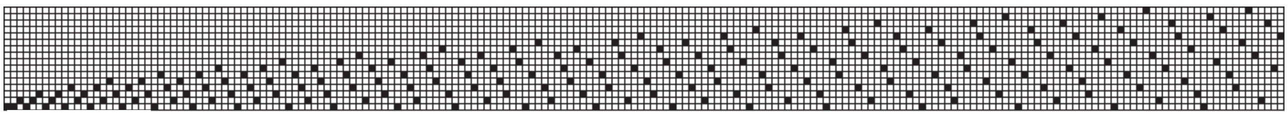
$\mathbb{S}(0.1)$



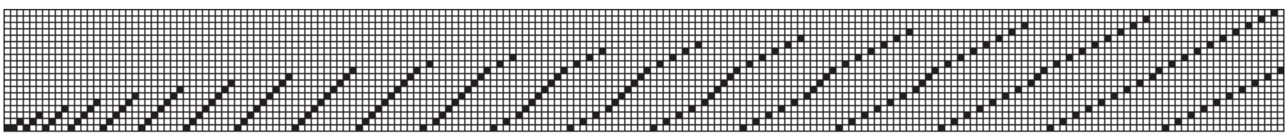
$\mathbb{S}(0.2)$



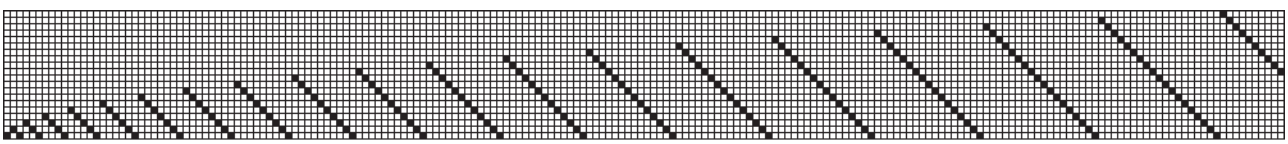
$\mathbb{S}(0.5)$



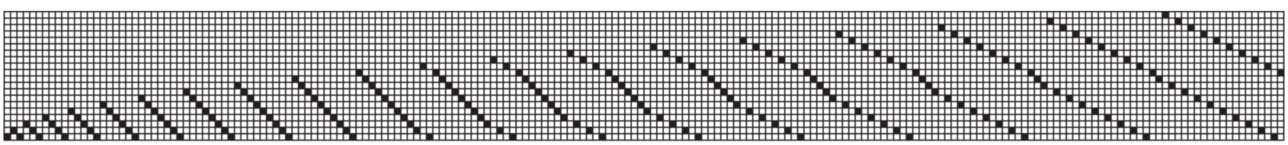
$\mathbb{S}(0.7)$



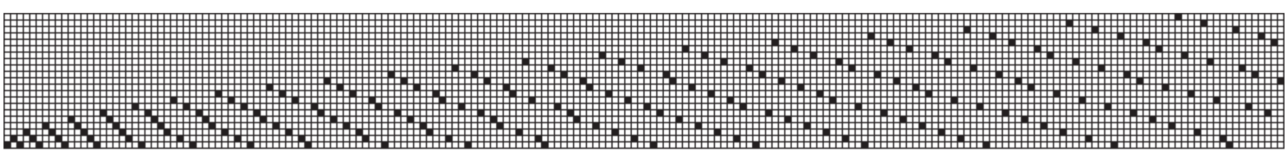
$\mathbb{S}(0.9)$



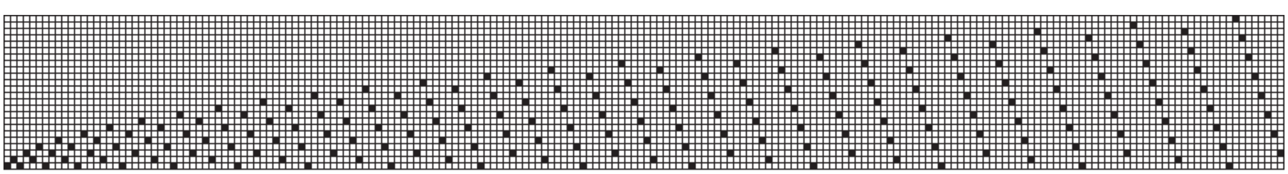
$\mathbb{S}(1.0)$



$\mathbb{S}(1.1)$

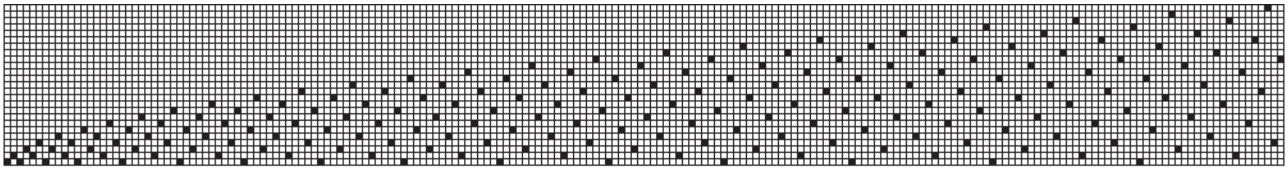


$\mathbb{S}(1.2)$

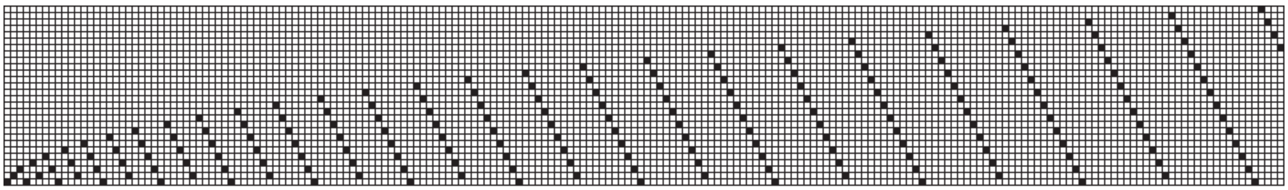


$\mathbb{S}(1.5)$

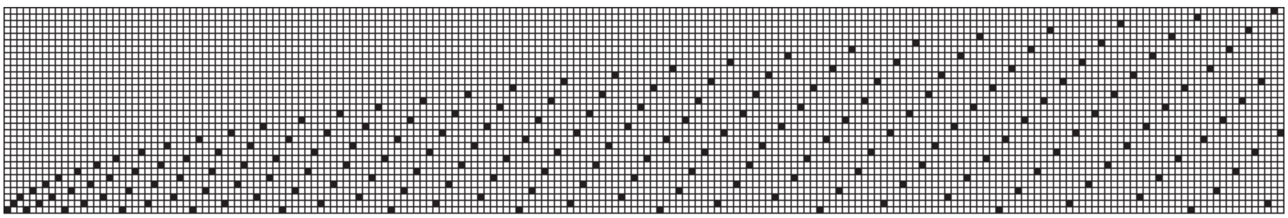
Figure 2. Grid Plots for Signature Sequences



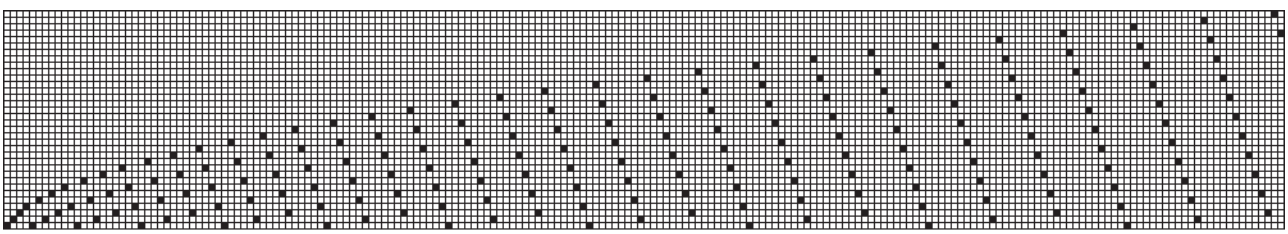
$\mathbb{S}(\phi)$



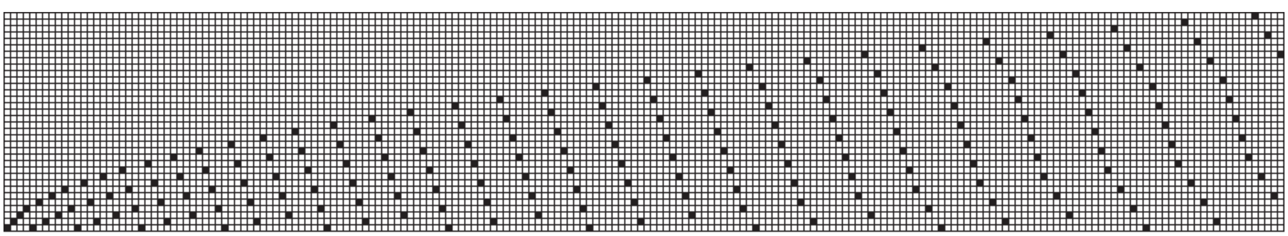
$\mathbb{S}(2.0)$



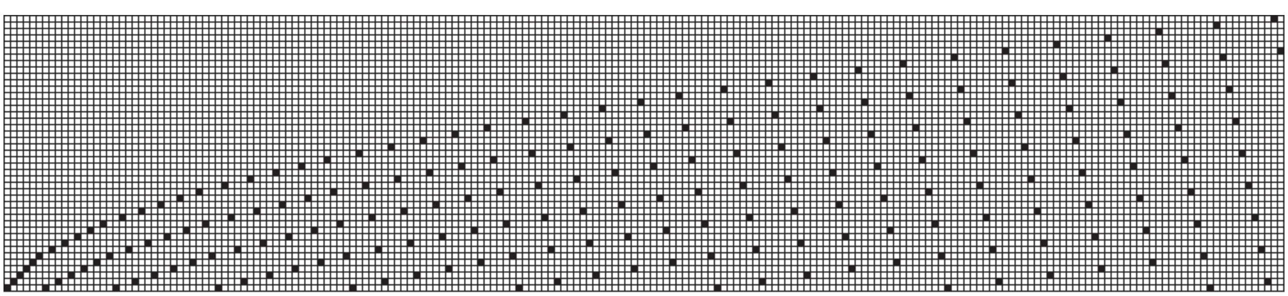
$\mathbb{S}(e)$



$\mathbb{S}(3.0)$

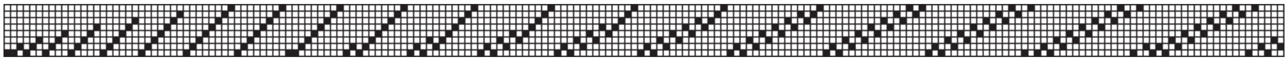


$\mathbb{S}(\pi)$

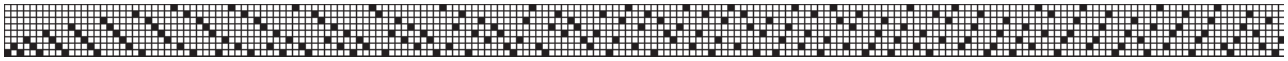


$\mathbb{S}(5.0)$

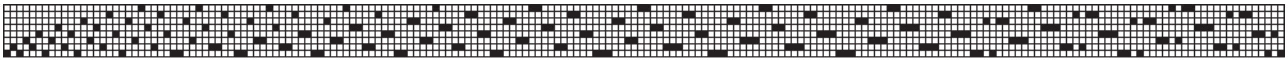
Figure 2, continued. Grid Plots for Signature Sequences



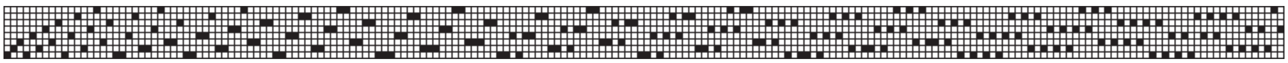
$\mathbb{S}(0.9)$ mod 8



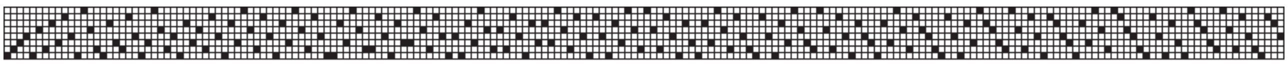
$\mathbb{S}(1.2)$ mod 8



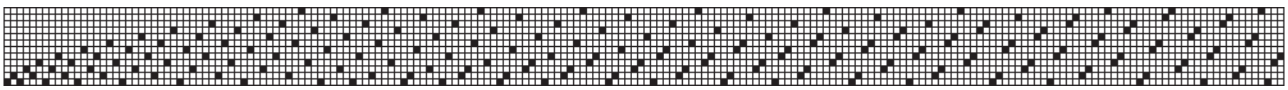
$\mathbb{S}(\phi)$ mod 8



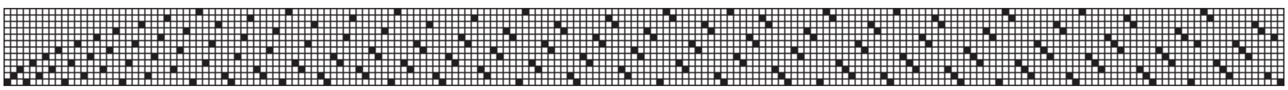
$\mathbb{S}(e)$ mod 8



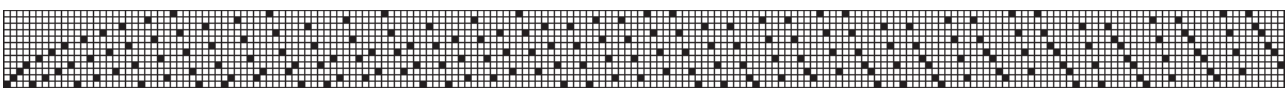
$\mathbb{S}(\pi)$ mod 8



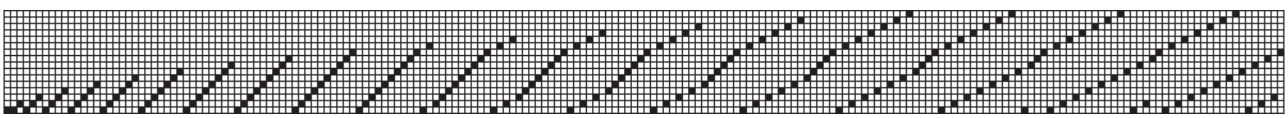
$\mathbb{S}(\phi)$ mod 12



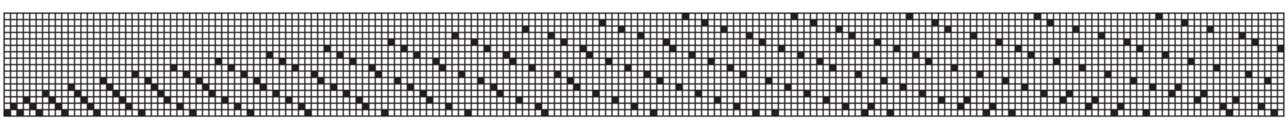
$\mathbb{S}(e)$ mod 12



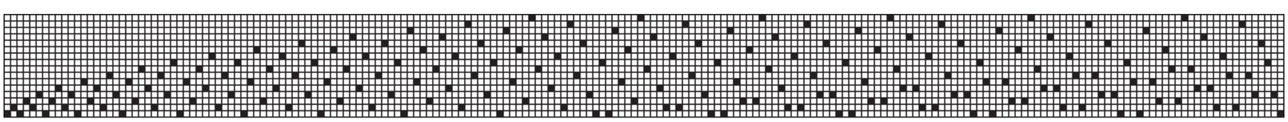
$\mathbb{S}(\pi)$ mod 12



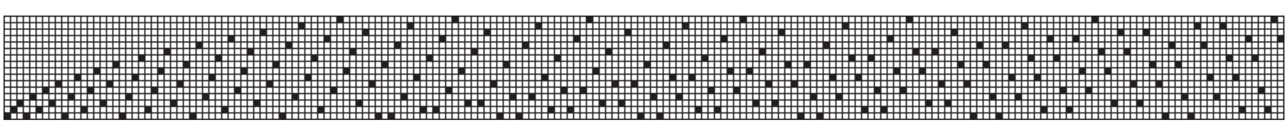
$\mathbb{S}(0.9)$ mod 16



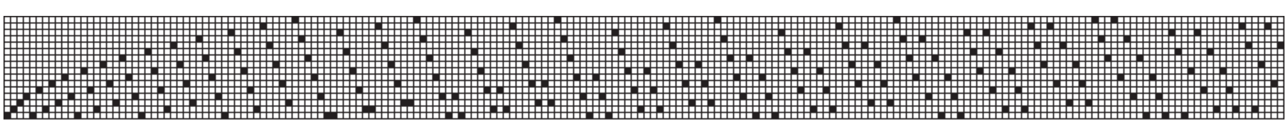
$\mathbb{S}(1.2)$ mod 16



$\mathbb{S}(\phi)$ mod 16



$\mathbb{S}(e)$ mod 16



$\mathbb{S}(\pi)$ mod 16

Figure 3. Signature Sequence Residues

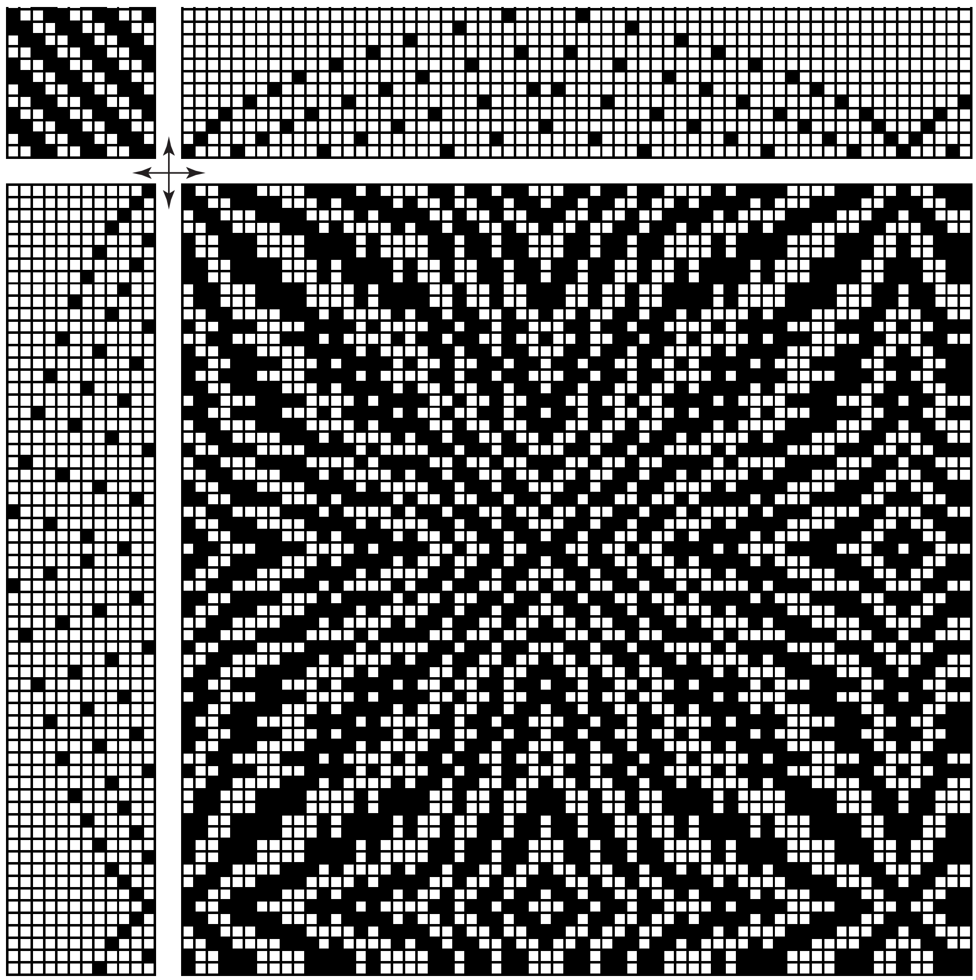


Figure 4. Drawdown for A Reflected Portion of $\mathbb{S}(\boldsymbol{\pi})$

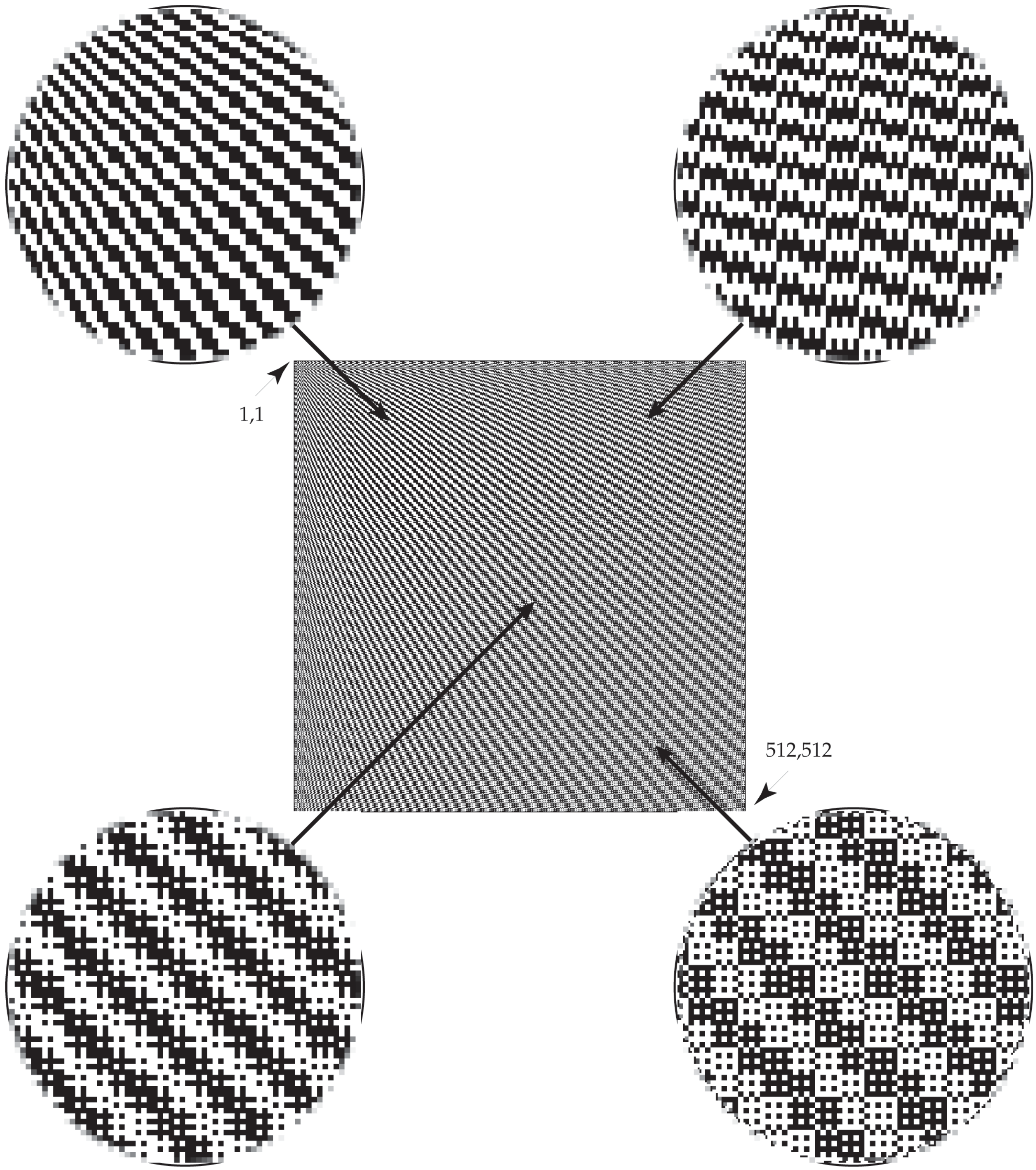
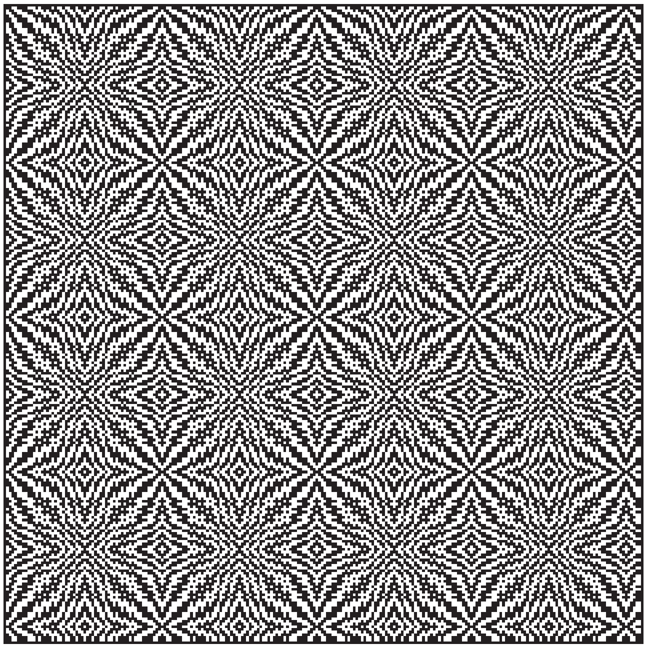
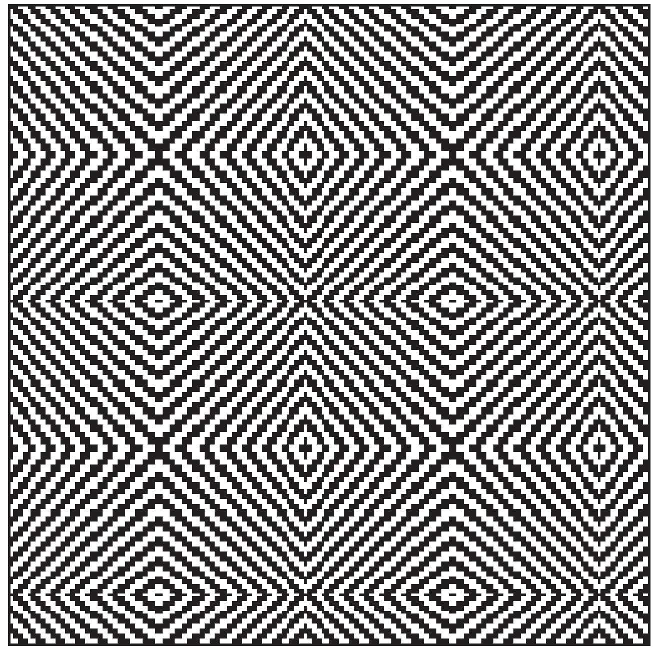


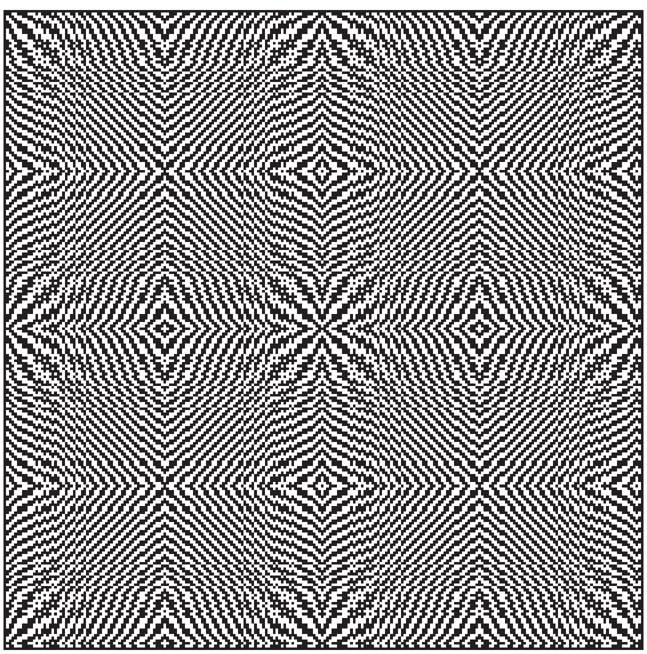
Figure 5. Magnified Portions of the ϕ×ϕ Signature Drawdown Plane



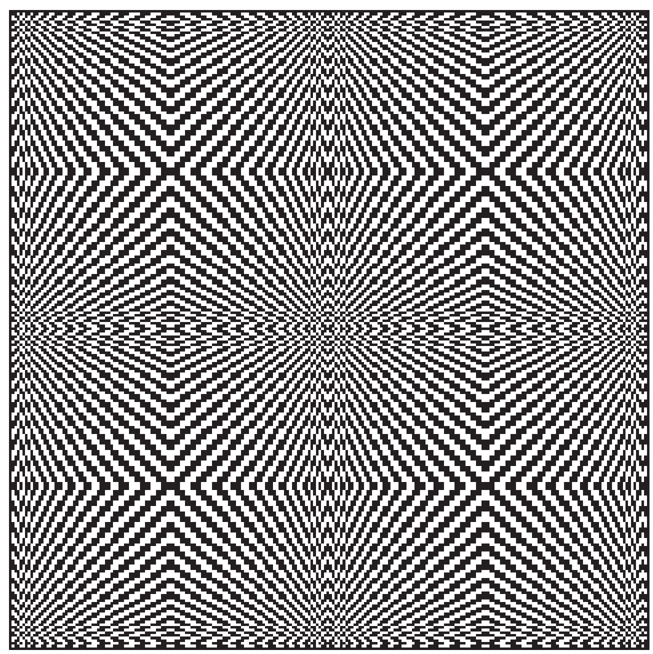
threading: π, terms 1-30
treadling: π, terms 1-30



threading: ϕ, terms 61-120
treadling: ϕ, terms 61-120

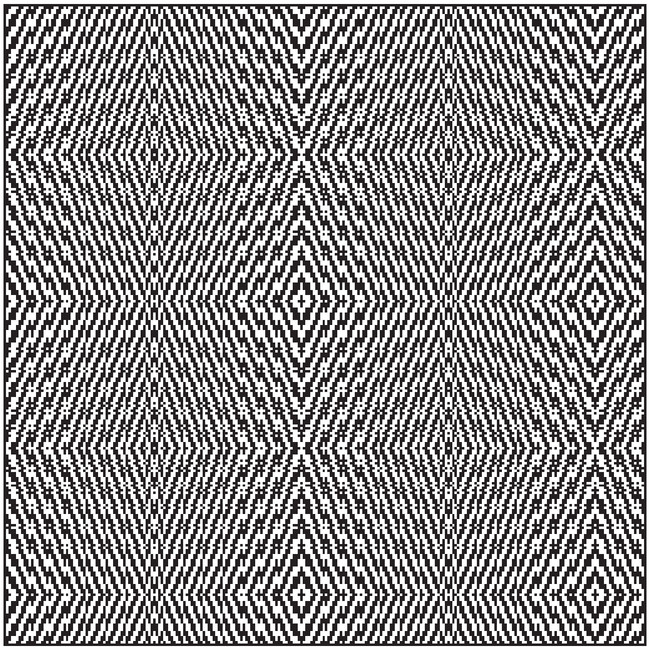


threading: π, terms 1-60
treadling: π, terms 1-60

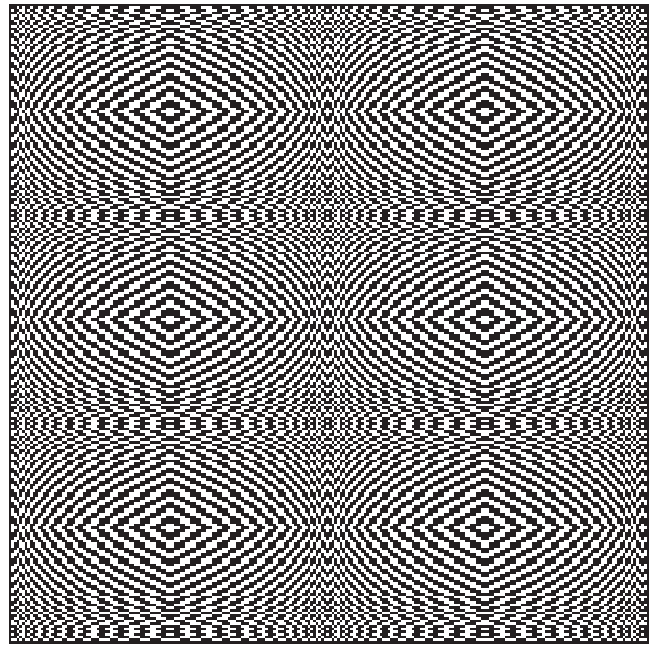


threading: *e*, terms 1-60
treadling: *e*, terms 1-60

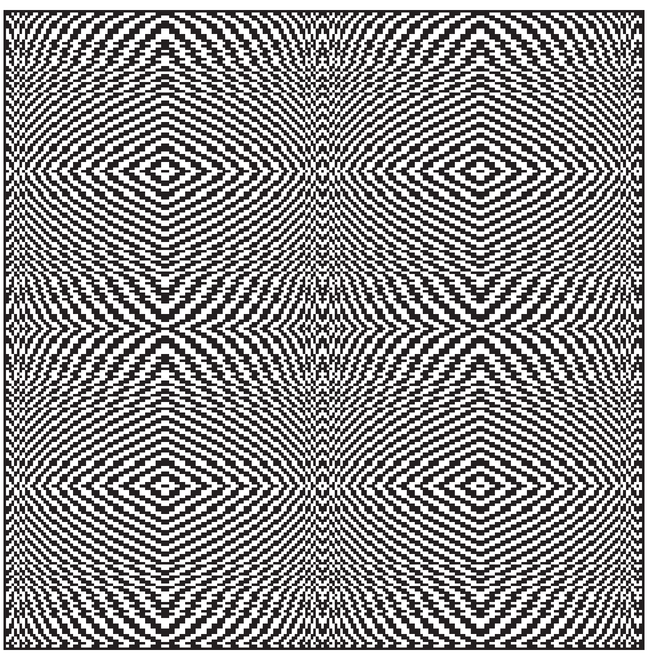
Figure 6. Drawdown Patterns for Signature Sequences



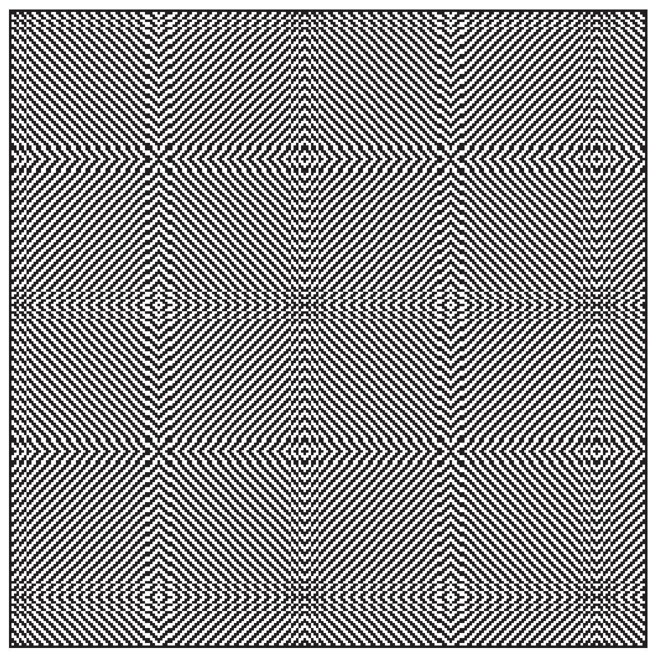
threading: π, terms 61-120
treadling: *e*, terms 61-120



threading: *e*, terms 1-60
treadling: $\sqrt{e}$, terms 1-40

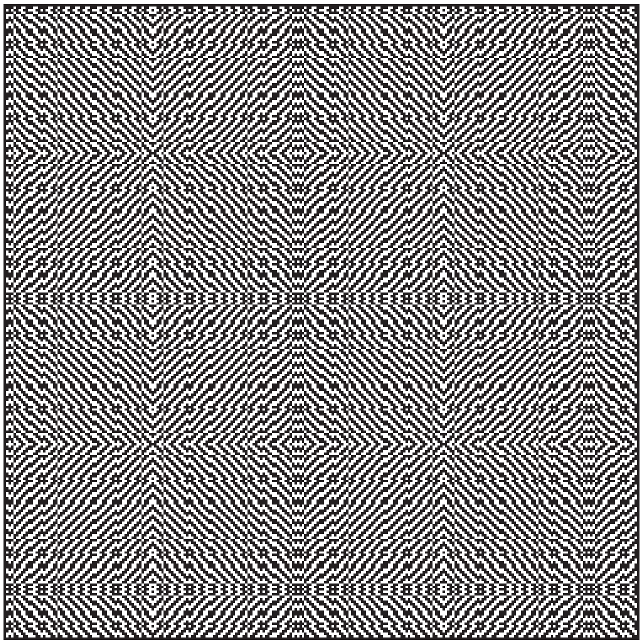


threading: *e*, terms 1-60
treadling: ϕ, terms 1-60

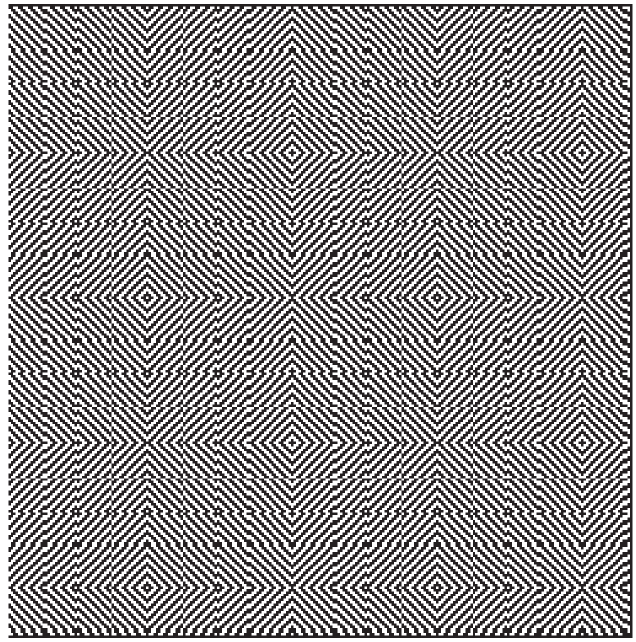


threading: 0.9, terms 61-120
treadling: 0.9, terms 61-120

Figure 6, continued. Drawdown Patterns for Signature Sequences



threading: 1.0, terms 1-60
treadling: 1.0, terms 1-60



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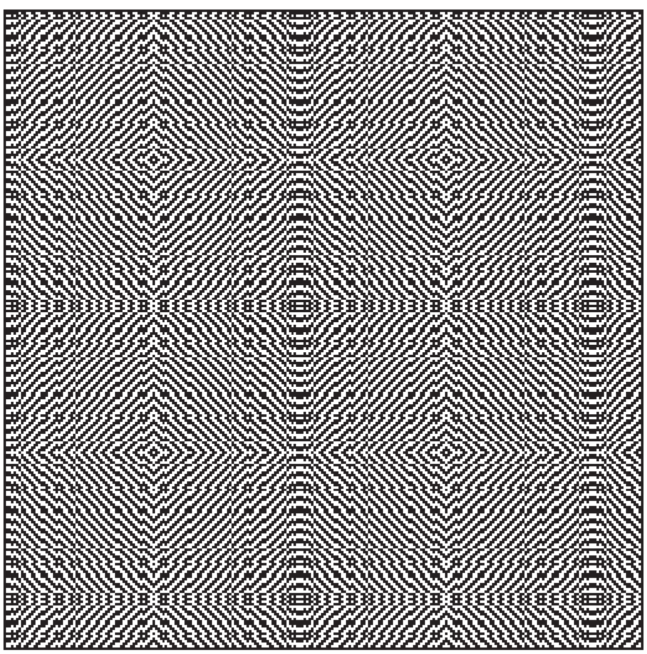
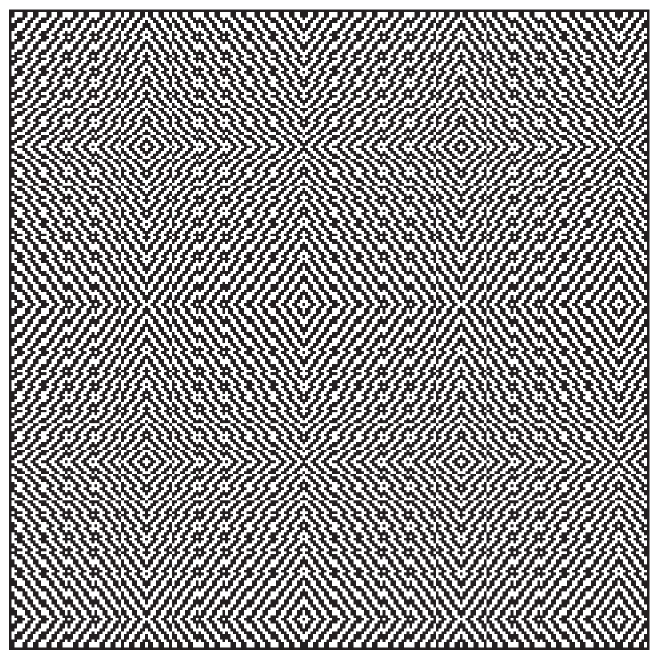
treadling: 1.0, terms 61-120



threading: 0.9, terms 1-60
treadling: 1.1, terms 1-60



threading: 0.5, terms 61-120
treadling: 1.5, terms 31-90

Figure 6, continued. Drawdown Patterns for Signature Sequences