Twill Counters

Twills are described by *counters* that show what shafts are raised and not raised.

Notation

Two notations are in common use for counters. In one, there is a horizontal line with pairs of numbers above and below, alternating, to show shafts raised and not raised. For example,

$$\frac{2 \quad 3}{2 \quad 1}$$

describes an 8-shaft twill counter in which the first two shafts are raised, the next two are not, the next 3 are, and the last, not.

This form of stacked notation is easy to understand, but it is typographically difficult to produce and requires setting apart from lines of text. An alternative, linear notation uses separating slashes to indicate the under/over sequence. In this notation, the example above would be written 2/2/3/1.

The difficulty with the linear notation is in keeping track of the over/under order. Nonetheless, the ease with which the linear notation can be written and used in text generally makes it the favored notation.

Twill Tie-Ups

In *regular twills*, the twill counter appears rotated by one for each successive row of the corresponding tie-up. Figure 1 shows the tie-up for the twill counter in the example above.
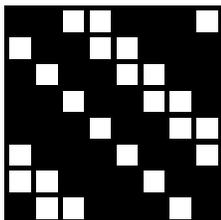
Figure 1. A 2/2/3/1 Twill Tie-Up

In this example, the rotation is to the right. This is called a *right twill* and is the one usually shown in examples. Rotation also can be to the left, producing a *left twill*.

As the twill counter is rotated to the right, parts of it move off the right end and onto the left end. The third row of the tie-up in Figure 1, taken as a twill counter, would look like this:

$$\frac{1 \quad 2 \quad 2}{1 \quad 2}$$

This is not a valid counter because there are more terms above the line than below. What has happened is that the 3 on the top of the original counter has been split into two parts: 2 at the end and 1 at the beginning.

The second row of the tie-up in Figure 1, on the other hand, would look like this as a counter:

$$\frac{2 \quad 3}{1 \quad 2}$$

This violates the convention that twill counters start with raised shafts.

The fourth row of the tie-up in Figure 1 corresponds, structurally, to a valid twill counter:

$$\frac{3 \quad 2}{1 \quad 2}$$

This counter is, in fact, equivalent to the original twill counter in our example:

$$\frac{2 \quad 3}{2 \quad 1}$$

The difference is that the 3/1/2/2 counter is a rotated version of the 2/2/3/1 counter.

Since rotation of a counter in multiples of two produces equivalent counters, the question is how to tell rotated counters apart, or better, how to pick a standard form.

Most authors pick the form that is, in a loose sense, the "smallest" — the one starting with the smallest number. Thus, 2/2/3/1 is smaller than 3/1/2/2. This is easier to see if the slashes are removed: 2231 is smaller than 3122.

In the case of counters with more parts, the standard one can be obtained by forming all rotations by multiples of two and picking the smallest of the results.

The Number of Twill Counters

There are no real twill counters for 2 shafts, although 1/1, which corresponds to plain weave, is an acceptable twill counter structurally. We'll include it in what follows so that we don't have to make exceptions for it constantly.

For 3 shafts, there are two twill counters, 1/2 and 2/1. Four 4 shafts, there are four: 1/1/1/1, 1/2, 2/2, and 3/1, 1/1/1/1 is simply a doubling of the 2-shaft 1/1.

As the number of shafts increases, it becomes increasingly difficult to figure out all the possible twill counters, especially if doing it by hand. In fact, mistakes can be found in this regard in old weaving books. For example, Posselt's *Technology of Textile Design* [1] omits some of the twill counters for 6 and 8 shafts.

Whether working by hand or using a computer, a systematic method is needed. And, as the number of shafts gets large, the method needs to be efficient.

A Method for Determining Twill Counters

The sum of the numbers in a twill counter add up to the number of shafts being considered. For example, for 4 is the sum of smaller numbers in six ways: $1 + 1 + 1 + 1$, $1 + 1 + 2$, $1 + 2 + 1$, $1 + 3$, $2 + 2$, and $3 + 1$. Of these, only the ones with an even number of terms correspond to twill counters: $1 + 1 + 1 + 1$, $1 + 3$, $2 + 2$, and $3 + 1$.

In mathematics, expressing a number as the sum of smaller (positive) numbers is called a *composition* [2]. In compositions, as in twill counters, order matters: $1 + 3$ is different from $3 + 1$. (If order doesn't matter, so that $1 + 3$ and $3 + 1$ are considered to be the same, these are called *partitions*.)

So the problem of finding twill counters is equivalent to the problem of finding compositions with even numbers of terms.

A convenient way to formulate the problem is to treat the number of shafts, n as a line with n segments of equal length, each segment amounting to one part of n. For $n = 6$, the line, with diamonds connecting the segments, looks like this:

Notice that there are five connection points, one less than the number of segments.

A composition can be obtained by selecting connection points. For example, the composition 2

$+ 2 + 1 + 1$ has the selected connection points as shown by the arrows:
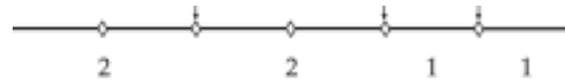
Notice for m segments, only $m - 1$ connection points are selected.

The composition, looked at in this way, can be represented by a bit pattern in which 0 means a connection point is not selected, and 1 means a connection point is selected. Therefore, the composition in the example above has the bit pattern 01011.

Note: It you're not interested in mathematics, just skip the next few paragraphs. They are not important.

In general, for n shafts, there are $n - 1$ connection points, 2^{n-1} possible patterns, and $2^{n-1} - 1$ compositions (since an all-0 pattern, which represents n itself, is not considered to be a composition). The number of m-segment compositions of n is given by the binomial coefficient

$$\binom{n-1}{m-1} = \frac{(n-1)!}{(n-m)! \times (m-1)!}$$

For twill counters, compositions with an even number of parts (an odd number of connection points) are required. The number of these is given by

$$\sum_{m \text{ odd}, < n} \binom{n-1}{m-1}$$

For example, for $n = 6$, the number of twill counters is

$$\binom{5}{5} + \binom{5}{3} + \binom{5}{1} = 1 + 10 + 5 = 16$$

Some of these consist of repeats of counters for a smaller number of shafts.

In terms of bit patterns, only those with 5, 3, or 1 1s correspond to twill counters. There are few enough of these to list them here:

11111	01101
11100	01011
11010	00111
11001	10000
10110	01000
10101	00100
10011	00010

```
   01110        00001
```

To convert binary pattern to a counter, proceed as follows:

1. Start at the left.

2. Remove 0s up to the first 1.

3. The corresponding counter number is the number of zeros removed + 1 (just 1 if there are no zeros).

4. Repeat steps 2 and 3 until there are no more 1s.

5. The last counter number is the number of 0s remaining plus 1 (just 1 if there are no remaining zeros).

As an example, consider the pattern 10110.

There are no zeros before the leftmost 1, so the first counter number is 1, and starting to construct the counter, we have $1/$.

What remains is 0110. Now there is one zero before the first 1, so the next counter number is 2, and the evolving counter is $1/2$.

What remains is 10. There are no 0s before the 1, so the next counter number is 1 and the evolving counter is $1/2/1$.

All that remains is 0. Since there are no more 1s, the last counter number is 2 (one remaining 0 plus 1), and the complete twill counter is $1/2/1/2$.

Note that if you just want counters with a specific number of counter numbers, you can use patterns that have that number of 1s less 1. For example, for 6-shaft counters with only four numbers (counters of the form $i/j/m/n$), you only need to decode

```
   11100        10101
   11010        10011
   11001        01110
   10110        00111
```

The method given here produces twill counters that are not in standard form that duplicate ones in standard form. For example,

$$00111 \rightarrow 3/1/1/1$$
$$11100 \rightarrow 1/1/1/3$$

Here the first twill counter is not in standard form but is a rotation of the second, which is in standard form.

It is necessary to remove twill counters that are not in standard form using the method described earlier. For 6 shafts, four of the counters are not in standard form, leaving a total of 12 twill counters:

```
   /1/1/1/1/1/1
   /1/1/1/3
   /1/1/2/2
   /1/1/3/1
   /1/2/1/2
   /1/2/2/1
   /1/5
   /2/1/2/1
   /2/4
   /3/3
   /4/2
   /5/1
```

Proper and Inherited Twill Counters

As mentioned above, some twill counters produced by this method may consist of repeats of twill counters for a smaller number of shafts. For example, $1/2/1/2$ from the worked-out example above is a repeat of the 3-shaft twill counter $1/2$.

Counters that are repeats of counters for a smaller number of shafts are called *inherited counters*. The rest are called *proper counters*.

There is nothing wrong with an inherited counter; it's just that it comes from a smaller number of shafts.

The number of inherited counters depends on the divisors of the number of shafts. For example, 6 has divisors 2 and 3. Thus, all of the (proper) counters for 2 shafts and 3 shafts are inherited for 6 shafts.

If a divisor itself has inherited counters, these are included in *its* divisors. For example, for 12 shafts, counters are inherited from 2-, 3-, 4-, and 6-shaft counters. The inherited 6-shaft counters are included in the proper 2- and 3-shaft counters, which are inherited for 12 shafts by virtue of its divisors 2 and 3.

How Many Twill Counters Are There?

The number of twill counters increases rapidly with the number of shafts.

Here are the numbers through 20 shafts:

shafts	proper	inherited	total
2	1	0	1
3	2	0	2

4	3	1	4
5	6	0	6
6	9	3	12
7	18	0	18
8	30	4	34
9	56	2	58
10	99	7	106
11	186	0	186
12	338	12	350
13	630	0	630
14	1161	19	1180
15	2182	8	2190
16	4080	34	4114
17	7711	0	7711
18	14543	57	14600
19	27594	0	27594
20	52377	109	52486

Note that if the number of shafts is a prime, there are no inherited twill counters because there are no divisors.

For more than 8 or so shafts, there are so many twill counters that it doesn't make sense to list them all — how could they all be used?

Acknowledgments

References

1. Posselt, E. A. *Technology of Textile Design*, Posselt, 1888.

2. Reingold, Edward M., Jurg Nievergelt, and Narsingh Deo, *Combinatorial Algorithms: Theory and Practice*, Prentice-Hall, 1977.

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