Hadamard Matrices and Weaving

1. Introduction

Hadamard matrices are a class of square matrices first described by James Sylvester (1814-1897) in 1867. He called them anallagmatic pavement. In 1893, Jacques Hadamard (1865-1963) discussed them in relation to what is now called Hadamard's theorem on determinants, and his name stuck. Hadamard matrices have several interesting properties and have found use in tessellation, signal processing, error detection and correction codes, statistics, combinetics, combinational block designs, and now, weaving.

2. Definitions

A Hadamard matrix, $\mathbf{H}_n$, is a square matrix of order n = 1, 2, or 4k[1] where k is a positive integer. The elements of $\mathbf{H}$ are either +1 or –1 and $\mathbf{H}_n\mathbf{H}_n^T = n\mathbf{I}_n$, where $\mathbf{H}_n^T$ is the transpose of $\mathbf{H}_n$, and $\mathbf{I}_n$ is the identity matrix of order n. A Hadamard matrix is said to be normalized if all of the elements of the first row and first column are +1.

3. Properties

Hadamard matrices have several interesting properties:

- The determinant, $|\mathbf{H}_n| = n^{n/2}$, is maximal by Hadamard's theorem on determinants.
- A normalized $\mathbf{H}_n$ has n(n-1)/2 elements of –1 and n(n+1) elements of +1.
- For normalized Hadamard matrices of order 2 or greater, every row (except the first) or column (except the first) has n/2 elements of +1 and n/2 elements of –1.
- Any two rows or two columns are orthogonal.
- Every pair of rows or every pair of columns differs in exactly n/2 places.

A Hadamard matrix may be transformed into an equivalent Hadamard matrix by any of the following operations:

- Interchanging any two rows or any two columns
- Multiplying any row or any column by –1
- Matrix transpose

Using these operations, it is possible to normalize any Hadamard matrix. The problem of determining if two Hadamard matrices are equivalent is very difficult.

The number of distinct (*i.e.*, non equivalent) Hadamard matrices for various orders is given in the Table 1. [Wal]

[1] This is conjectured to be true for all k, and has been verified for n < 688. There are several methods for the construction of Hadamard matrices [Wal], several of them for n of particular forms.

Order	1	2	4	8	12	16	20	24
Number of distinct Hadamard matrices	1	1	1	1	1	5	3	60

Table 1. Number of Distinct Hadamard Matrices for Various Orders

4. Examples

In order to more easily visualize the Hadamard matrices, let us map the +1 elements as white squares, and the –1 elements as black squares. Figure 1 show normalized Hadamard matrices for the orders of n = 1, 2, 4, 8, 12, and 16.

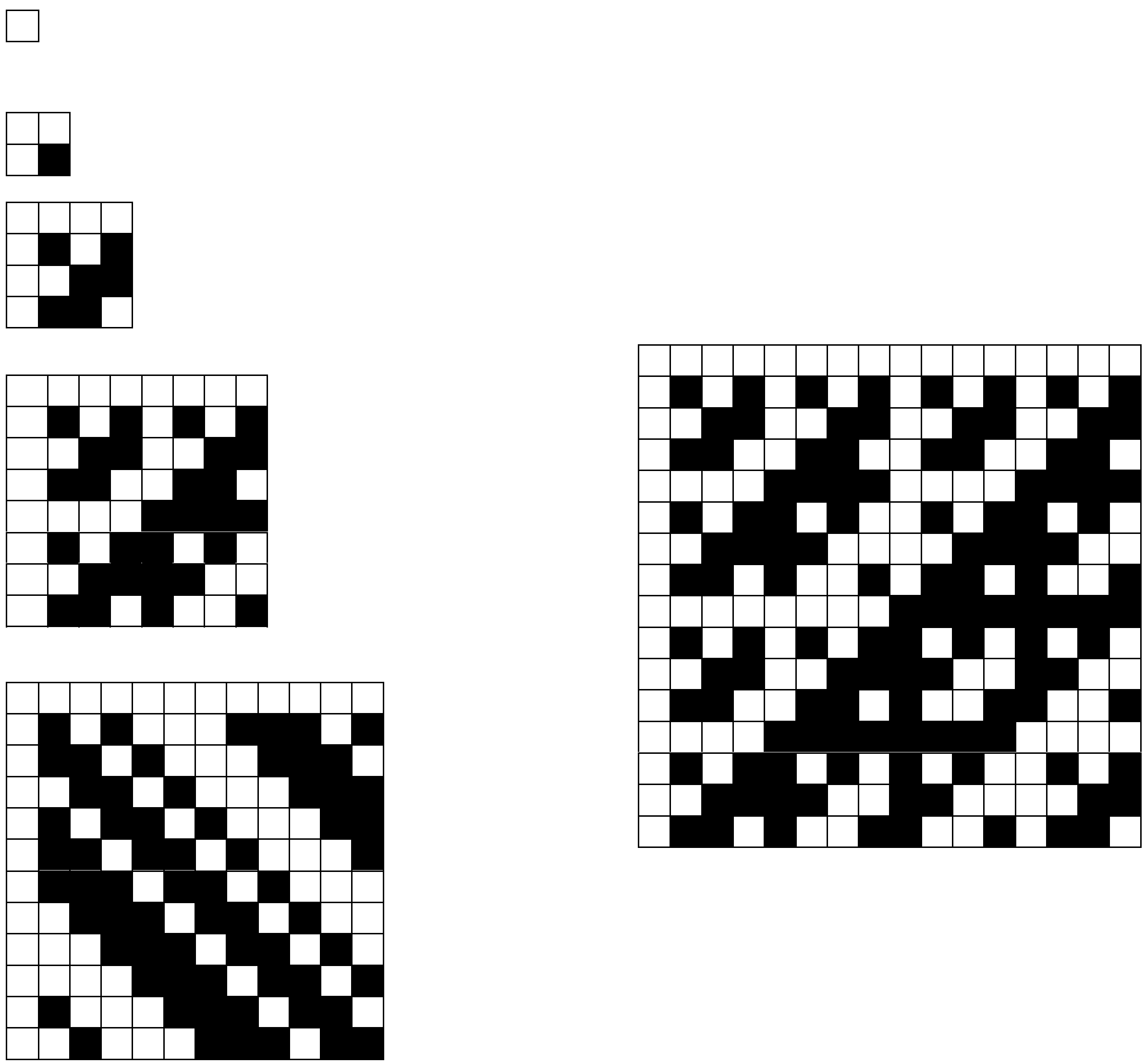

Figure 1. Hadamard matrices of order n = 1, 2, 4, 8, 12, and 16

5. Relevance to Weaving

If we consider a Hadamard matrix to be a tie-up, then many of the properties described in section 3 such as the near balance of +1 and –1 elements, a bound number of difference between rows, and orthogonality would lead one to believe that it would produce a good weave. Indeed, the normalized $\mathbf{H}_{12}$ above is an 11-shaft twill tie-up. Using the operations in section 3, it is relatively easy to create tie-ups. Using $\mathbf{H}_4$ in figure 2(a) to demonstrate, multiply the first column by -1 giving the result in figure 2(b). Then multiply the first row by –1 to give figure 2(c). Then by interchanging rows 2 and 3, we end up with a warp surface twill tie-up in figure 2(d)[2].

(a) **(b)** **(c)** **(d)**

Figure 2. Transformation of $\mathbf{H}_4$ into a tie-up

A more complex transformation of $\mathbf{H}_{16}$ is shown in figure 3 as a tie-up and possible drawdown on the next page.

6. Conclusion

Even though Hadamard matrices are conceptually simple, they have some surprising properties and uses. For weaving, Hadamard matrices provide an interesting design foundation for tie-ups.

7. References

[Hed] Hedayat, A., Wallis, W. D., *Hadamard Matrices and their Applications*, The Annals of Statistics, Vol. 6, Number 6, pp 1184-1238, 1978.

[Wal] Wallis, W.D., Street, A. P., Wallis, J. S., *Combinatorics: Room Squares, Sum-free Sets, Hadamard Matrices*, Lecture Notes in Mathematics, Vol. 292, Springer-Verlag, New York, 1972.

[2] It is conjectured that this is the only circulant Hadamard matrix. Because of its relationship to Barker sequences, it is known to be the only one for n < 12,100. [Hed]

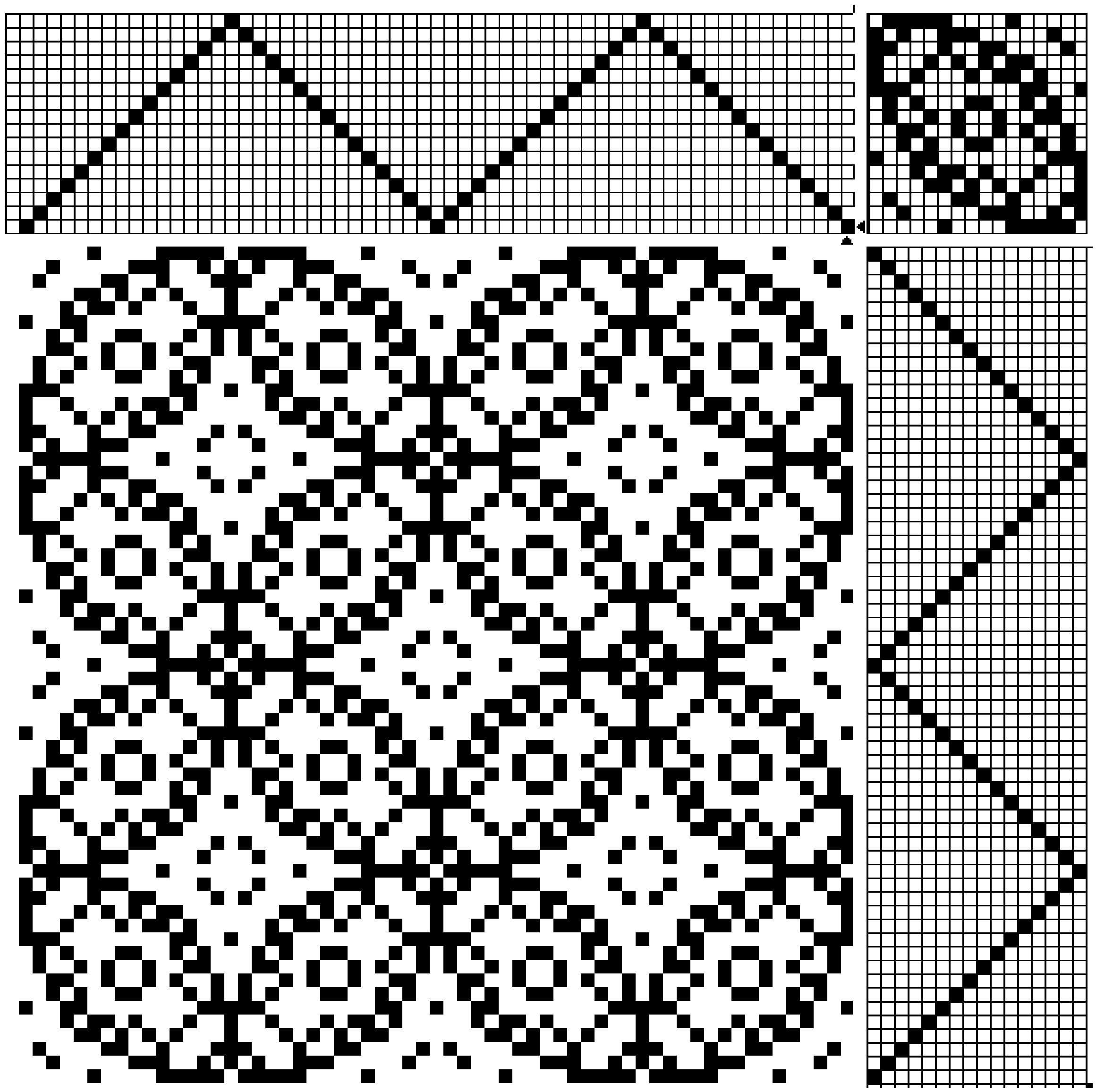

Figure 3. H_{16} Tie-up and Drawdown

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December 2004